ST. ANNE'S COLLEGE OF ENGINEERING AND TECHNOLOGY
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ANGUCHETTYPALAYAM, PANRUTI – 607 106.

DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING

NOTES

III SEMESTER

EC3351 – CONTROL SYSTEM

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EC3351 : CONTROL SYSTEMS ENGINEERING

II yr ECE

UNIT:1 SYSTEM COMPONENTS & THEIR REPRESENTATION

Control system: Terminologies & Basic structure - Feed Back & Feed Forward control theory - Electrical and Mechanical Transfer function models - Block diagram models - signal flow graph models - DC & AC servo motors - synchros - Multivariable control systems.

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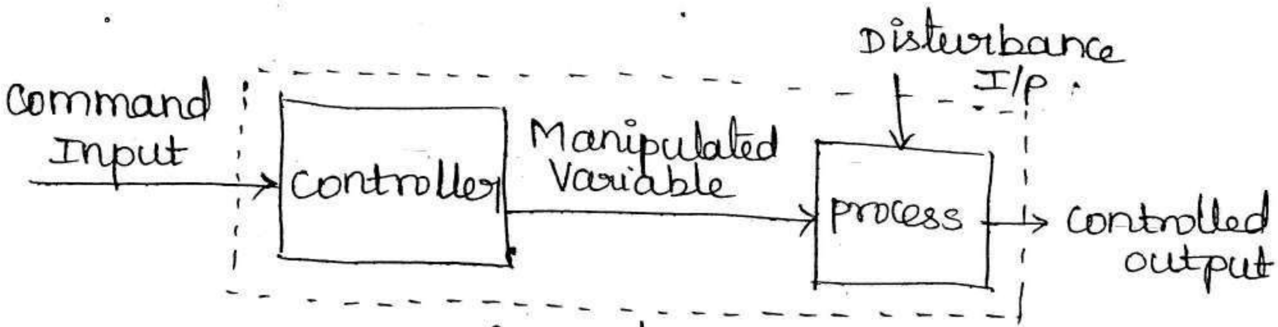
INTRODUCTION:

"Control systems" deal with control of Engineering systems that are governed by the laws of physics and are therefore called "Physical systems".

The word "control" means to regulate, to direct, or to command. The word "system" means a combination of devices and components connected together to perform a certain function. This system may be physical, biological, economic etc.,

"control system" is defined as combination of devices and components connected or related so as to command, direct or regulate it self or another system. It is used in many applications eg. control of temperature, liquid level, position, Velocity, flow, pressure, acceleration etc.,

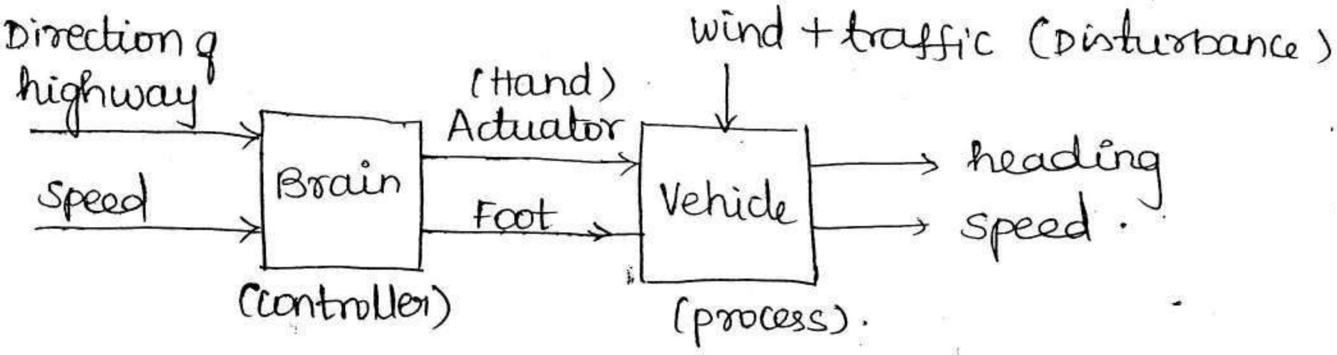
Configuration of a control system



Control system. (open loop)

eg. Automobile Driving system.

It has two inputs and two controlled outputs. Command inputs are direction of highway and speed limit with traffic signals. It can be represented as,



Control system Terminology :-

(i) Reference Input : It provides input signal for desired output

(ii) Error Detector : It is an element in which one system variable is subtracted from another variable to obtain third variable. also called comparator.

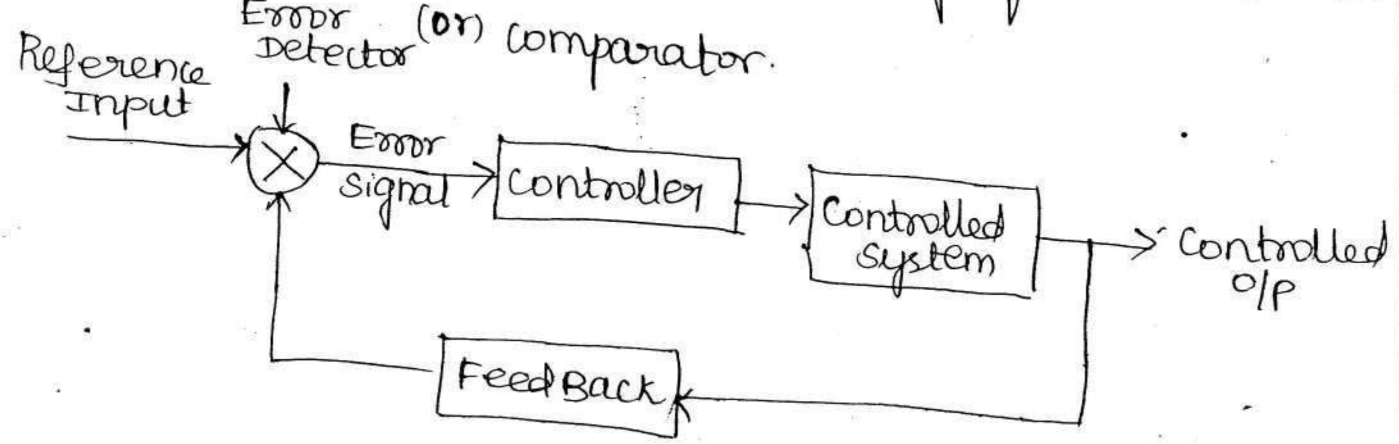
(iii) Feedback element : It measures the controlled output, convert or transforms to a suitable value for comparison with ref. Input.

(iv) <u>Error signal</u>: It is an algebraic sum of reference input and feedback.

(v) <u>controller</u>: controller is an element that is required to generate the appropriate control signal. The controller operates until the error between controlled output and desired output is reduced to zero.

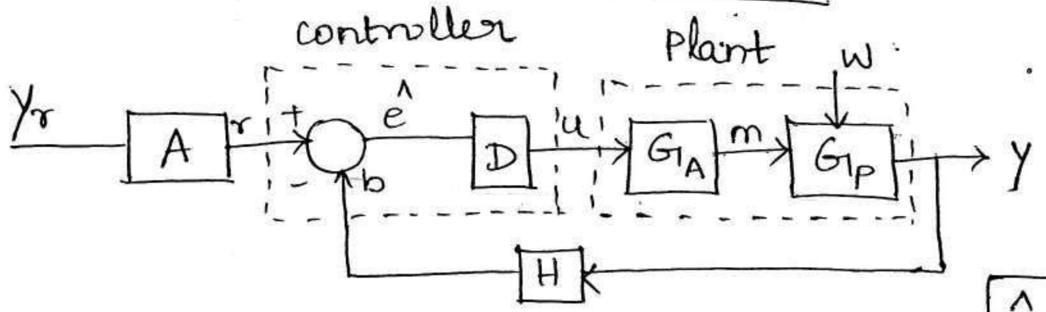
(vi) <u>controlled system</u>:- It is a body, a plant, a process or a machine of which a particular condition is to be controlled. for eg., room heating system, spacecraft, reactor boiler.

(vii) <u>controlled output</u>: It is produced by actuating signal available as input to the controller. controlled output is made equal to desired output with the help of feedback S/m.



"Representation of control system" (closed loop)

Basic structure of control system:



$\boxed{\hat{e} = r - b}$

Elements :-

A → Reference input element
D → control logic elements
G_A → Actuator elements
G_p → Controlled system elements.
H → feedback elements
b → feedback signal.

Y → controlled output
w → Disturbance input
$\hat{e}$ → Actuating error
u → control signal.
m → manipulated Variable
r → reference input
Y_r → command input.

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Classification of control systems

(i) Open and closed loop control system.
(ii) linear and non linear Control system
(iii) Time invariant and Variant control system
(iv) continous time & Discrete control system
(v) SISO & MIMO control system
(vi) lumped & Distributed parameter control system
(vii) Deterministic & Stochastic control system.
(viii) Static & Dynamic control system.

(i) open loop and closed loop systems.

open loop	closed loop.
No feedback used.	Feedback is used for compare the desired output and reference input
open loop system is generally stable.	closed loop system become unstable under certain conditions
simple to develop and cheap. Accuracy is determined by calibration of their elements	It is more complex. complicated to construct and costly.
Affected by non linearities.	Adjust to effect of non linearity present in the s/m.
eg., washing machine, fixed time traffic, room heater	eg. refrigerator, servo motor, smart AC, generator o/p.

(ii) <u>linear and non linear control system:-</u>

If a system obeys the principle of superposition. such a system is called linear system. The superposition principle states that response produced by simultaneous application of two different forcing function is equal to sum of individual responses.

If a system donot obey the superposition principle is called non linear system.

(iii) <u>Time invariant and Time Varying control system</u>

If system parameters donot Vary with time.

applied. such a system is called Time invariant.
eg. R, L, C.

If a system parameters vary with time. The response depend on time at which input is applied such a system is called Time Variant system.
eg., Space Vehicle control system. where mass decreases with time.

(iv) Continous time & Discrete control system:

If all system parameters are function of continous time 't'. is called Continous time control System: eg., Speed control of DC Motor.

If the system control involves one or more variables that are known only at discrete instants of time is called Discrete time control system.
eg. A/D converters, generator excitation Control system.

(v) SISO & MIMO:

A system with one command input and one controlled output is called Single Input single output [SISO].

A system with multiple inputs & outputs is called Multi Input - Multi output (MIMO).
eg. boiler drum level, robot arm control.

(vi) Lumped and Distributed parameter control system:-

If control systems described by ordinary Differential equations are lumped parameter. If control system described by partial Differential equations are called distributed parameter. eg. R, L, C.

(vii) Deterministic and Stochastic Control system:-

If the response is predictable and repeatable is called Deterministic. If response involve random Variable parameters. Such a system is called Stochastic control System.

(viii) Static and Dynamic control system:-

If present output depends on past input is called dynamic or time dependent system. If present output depends only on present input is called static or time independent system.

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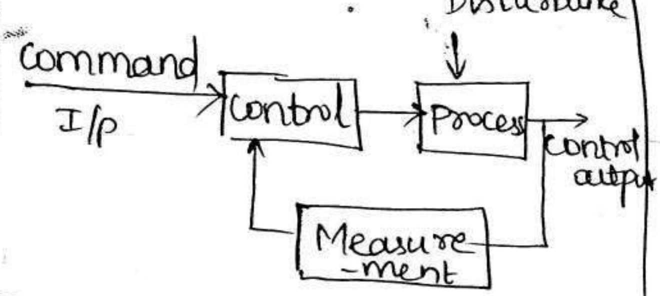
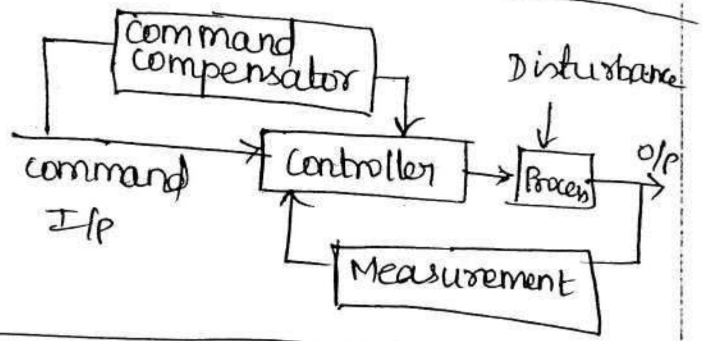
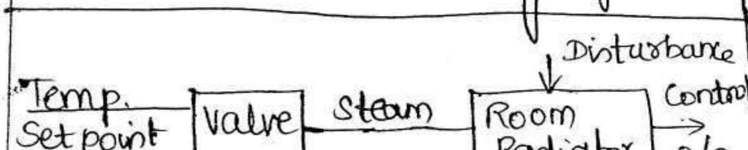
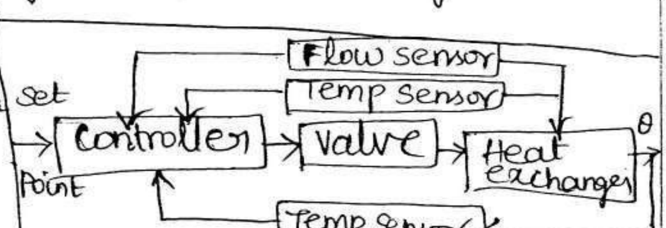
Feed Back & Feed Forward control theory:-

Both control technique is used to compensate the disturbance. It does not generally require any Specialized control theory. basic

Feed Back	Feed Forward
	
Main objective is to error Self nulling.	Main objective is to control or minimize transient error
. It compensates the any disturbance which affecting controlled variable.	It compensate any disturbance before they affect the controlled variable
It does not include Feed Forward structure.	It includes Feedback structure
once disturbance enters a process, it must propage through process and force the controlled variable to deviate before corrective actions are taken	Disturbance are measured before they enter the process and required value of manipulated variable to maintain desired controlled Variable.
eg. Residential Heating system	eg. Heat Exchanger.
	

Electrical & Mechanical Transfer function Models:

In control theory, transfer functions are commonly used to characterize the input-output relationship of components or systems that can be described by linear time-invariant differential equations.

Transfer function of a linear Time invariant system is defined as the ratio of Laplace Transform of the output to the Laplace transform of the input, under the assumptions that all initial conditions are zero.

$$\boxed{G(S) = \frac{C(S)}{R(S)}\Bigg|_{\text{zero initial conditions}}}$$

Let us consider a linear, Time invariant system defined by following differential equation:

$$[a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y + a_n] =$$
$$[b_0 x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m]$$

where $n \geq m$,

$y \rightarrow$ output of s/m

Taking Laplace Transform, on both sides,

$$(a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n) Y(s) =$$

$$(b_0 s^m + b_1 s^{m-1} + \cdots b_{m-1} s + b_m) X(s).$$

Transfer function : $G(s) = \left.\dfrac{L(\text{output})}{L(\text{input})}\right|_{\text{initial condition} = 0}$.

$$\boxed{G(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}.$$

using the concept of Transfer function,

(i) calculate $\boxed{\text{order}}$ of system by knowing highest. power of s in denominator.

(ii) calculate $\boxed{\text{Type}}$ of system by number of open-loop poles at the origin.

Always order of system is $\geq$ its type

$$\boxed{\text{order} \geq \text{type}}.$$

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1. The Transfer function of a system is given by

$$G(s) = \frac{K(s+6)}{s(s+2)(s+5)(s^2+7s+12)}$$. Determine

a) zeros, (b) poles, (c) characteristic eqn. (iv) pole-zero

Solution :-

(i) <u>zeros:</u>

Numerator terms equal to zero.

$\therefore\ s+6=0$

$$\boxed{s=-6}$$

(ii) <u>poles:</u>

Denominator terms equal to zero.

$$s(s+2)(s+5)(s^2+7s+12)=0$$

$$s(s+2)(s+5)(s+4)(s+3)=0$$

$$\therefore\ \boxed{s=0,-2,-5,-4,-3}$$

(iii) <u>characteristic equation:</u>

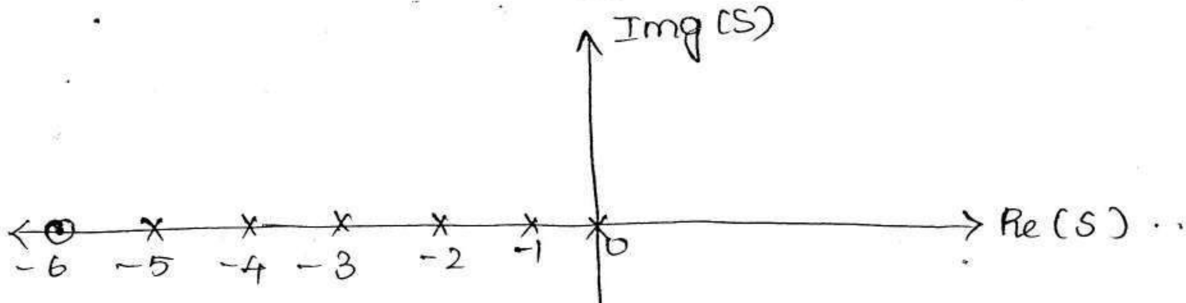
$$s(s+2)(s+5)(s^2+7s+12)=0$$

$$\Rightarrow (s^2+2s)(s^3+7s^2+12s+5s^2+35s+60)=0$$

$$\Rightarrow s^5+5s^4+7s^4+35s^3+12s^3+60s^2+2s^4+10s^3+14s^3+70s^2+24s^2+120s=0$$

$$\Rightarrow \boxed{s^5+14s^4+71s^3+154s^2+120s=0}$$

(iv) <u>pole-zero Diagram:</u>

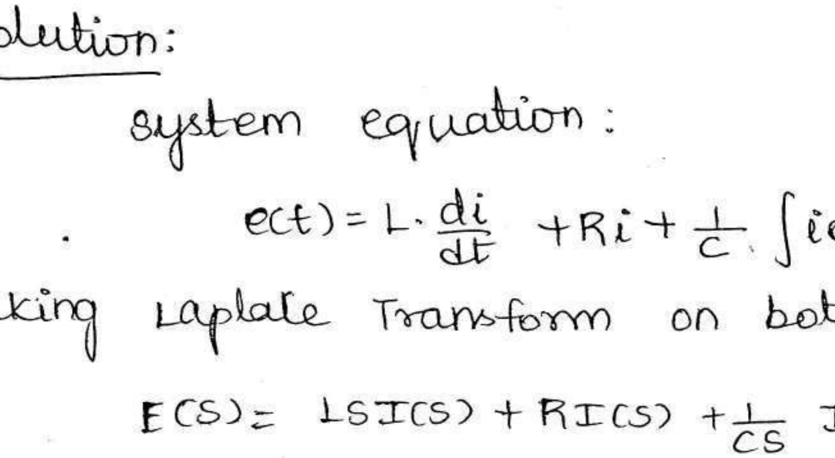


MODELLING OF ELECTRICAL SYSTEM:

Resistor, Inductor, capacitor are the three basic elements of an electric circuit. The circuit is analysed by application of Kirchhoff Voltage and current laws.

Sno	Element	Symbol	Time domain $V(t)=$	Laplace Domain $V(S)=$
1.	Resistor	$i(t)$ R, + $V(t)$ −	$i(t) \times R$	$R\,I(S)$
2.	Inductor	$i(t)$ L, + $V(t)$ −	$L \cdot \frac{d\,i(t)}{dt}$	$LS\,I(S)$
3.	Capacitor	$i(t)$ C, + $V(t)$ −	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{SC}\,I(S)$

1. Consider an electrical system of RLC series circuit, Find out Transfer function:



⇒ solution:

system equation:

$$e(t) = L \cdot \frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt \longrightarrow (1) \cdot \text{(Apply KVL)}$$

Taking Laplace Transform on both sides.

$$E(S) = LS\,I(S) + R\,I(S) + \frac{1}{CS}\,I(S).$$

$E(S) = [LS + R + \ldots$

Assuming all initial conditions to be zero.

$$E(s) = \left[Ls + R + \frac{1}{Cs}\right] I(s)$$

$$\boxed{E(s) = \left[\frac{LCs^2 + RCs + 1}{Cs}\right] I(s)} \longrightarrow (3)$$

Let the output Voltage $V_o(t)$ be taken across the capacitor, C. Then,

$$V_o(t) = \frac{1}{C}\int i\,dt \longrightarrow (4)$$

Taking Laplace Transform on both sides,

$$\boxed{V_o(s) = \frac{1}{Cs} I(s)} \longrightarrow (5)$$

∴ Transfer function is given by,

$$G(s) = \frac{V_o(s)}{E(s)} = \frac{\cancel{I(s)}/C(s)}{\left[(LCs^2 + RCs + 1)/Cs\right]\cancel{I(s)}}$$

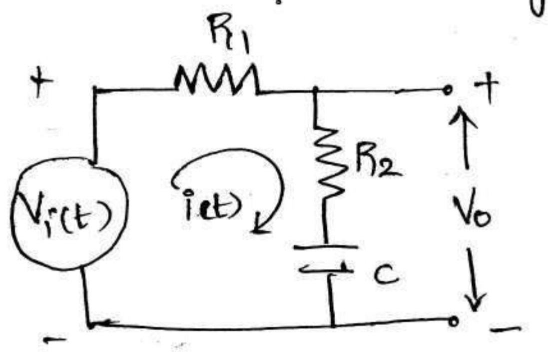
$$\boxed{G(s) = \frac{V_o(s)}{E(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Block Diagram Representation:-

$$E(s) \longrightarrow \underbrace{\boxed{\frac{1}{LCs^2 + RCs + 1}}}_{G(s)} \longrightarrow V_o(s)$$

This is second order

2. Determine the transfer function of electrical System as shown in Figure.



= Solution:

Loop equation :- (Apply KVL)

$$V_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)dt \longrightarrow ①$$

Taking Laplace Transform on both sides.

$$V_i(S) = R_1 I(S) + R_2 I(S) + \frac{1}{CS} I(S).$$

$$\boxed{V_i(S) = \left[R_1 + R_2 + \frac{1}{CS}\right] I(S)} \longrightarrow ②$$

Output equation:-

$$V_o(t) = R_2 i(t) + \frac{1}{C}\int i(t)\, dt \longrightarrow ③$$

Taking Laplace Transform on both sides.

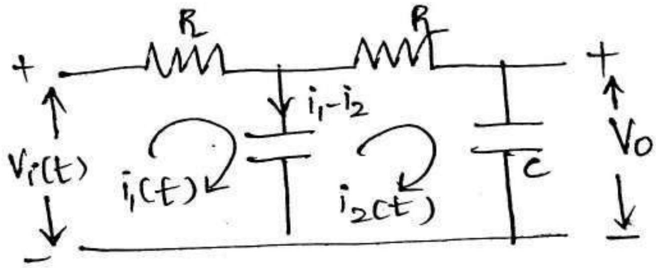
$$V_o(S) = R_2 I(S) + \frac{1}{CS} I(S).$$

$$\boxed{V_o(S) = \left[R_2 + \frac{1}{CS}\right] I(S)} \longrightarrow ③$$

Transfer function: $\frac{V_o(S)}{V_i(S)} = \left[R_2 + \frac{1}{CS}\right] \cancel{I}(S)$

$$\boxed{T.F = \frac{V_o(s)}{V_i(s)} = \frac{R_2 Cs + 1}{R_1 Cs + R_2 Cs + 1}}$$

3. Obtain the transfer function for electrical network.



= Solution:

Apply KVL to loop 1:

$$V_i(t) = R i_1(t) + \frac{1}{C}\int [i_1(t) - i_2(t)]\, dt \longrightarrow (1)$$

Taking Laplace Transform on both sides.

$$V_i(s) = R I_1(s) + \frac{1}{Cs}[I_1(s) - I_2(s)]$$

$$V_i(s) + \frac{1}{Cs} I_2(s) = \left[R + \frac{1}{Cs}\right] I_1(s)$$

$$I_1(s) = \frac{[V_i(s)\, Cs + I_2(s)] / \cancel{Cs}}{[1 + RCs] / \cancel{Cs}}$$

$$\boxed{I_1(s) = \frac{V_i(s)\, Cs + I_2(s)}{1 + RCs}} \longrightarrow (2)$$

Apply KVL to loop 2,

$R i_2(t) + \frac{1}{C}\int i_2(t)\, dt + \frac{1}{C}\int [i_2(t) - i_1(t)]\, dt$

Taking Laplace Transform on both sides,

$$R I_2(s) + \frac{1}{Cs} I_2(s) + \frac{1}{Cs}[I_2(s) - I_1(s)] = 0$$

$$I_2(s)\left[R + \frac{1}{Cs} + \frac{1}{Cs}\right] - \frac{1}{Cs} I_1(s) = 0$$

$$I_2(s)\left(\frac{2+RCs}{Cs}\right) = \frac{1}{Cs} I_1(s) \longrightarrow (4)$$

Substitute $\boxed{I_1(s) = \frac{V_i(s)\,Cs + I_2(s)}{1+RCs}}$

$\therefore$ Eq (4) becomes,

$$I_2(s)\left(\frac{2+RCs}{Cs}\right) = \frac{1}{Cs}\left[\frac{V_i(s)\,Cs + I_2(s)}{1+RCs}\right]$$

$$I_2(s)\left(\frac{2+RCs}{Cs}\right) = \frac{V_i(s)}{1+RCs} + \frac{I_2(s)}{Cs(1+RCs)}$$

$$I_2(s)\left[\frac{2+RCs}{Cs} - \frac{1}{Cs(1+RCs)}\right] = \frac{V_i(s)}{1+RCs}$$

$$I_2(s)\left[\frac{(2+RCs)(1+RCs)-1}{Cs\,\cancel{(1+RCs)}}\right] = \frac{V_i(s)}{\cancel{1+RCs}}$$

$$\boxed{I_2(s) = \frac{V_i(s)\,Cs}{1+3RCs+R^2C^2s^2}}$$

$$V_i(s) = \frac{I_2(s)\left[1+3RCs+R^2C^2s^2\right]}{} \longrightarrow (5)$$

Output equation:- $V_o(t) = \frac{1}{C}\int i_2(t)\,dt \rightarrow (6)$

Taking Laplace Transform,

$$\boxed{V_o(s) = \frac{1}{CS} I_2(s)} \rightarrow (7).$$

Transfer function: $G(s) = \frac{V_o(s)}{V_i(s)} = \frac{\cancel{I_2(s)}/\cancel{CS}}{\cancel{I_2(s)}(1+3RCS+R^2C^2S^2)/\cancel{CS}}$

$$\therefore \boxed{G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1+3RCS+R^2C^2S^2}}$$

Block Diagram Representation:-

$$V_i(s) \longrightarrow \boxed{G(s) = \frac{1}{1+3RCS+R^2C^2S^2}} \longrightarrow V_o(s).$$

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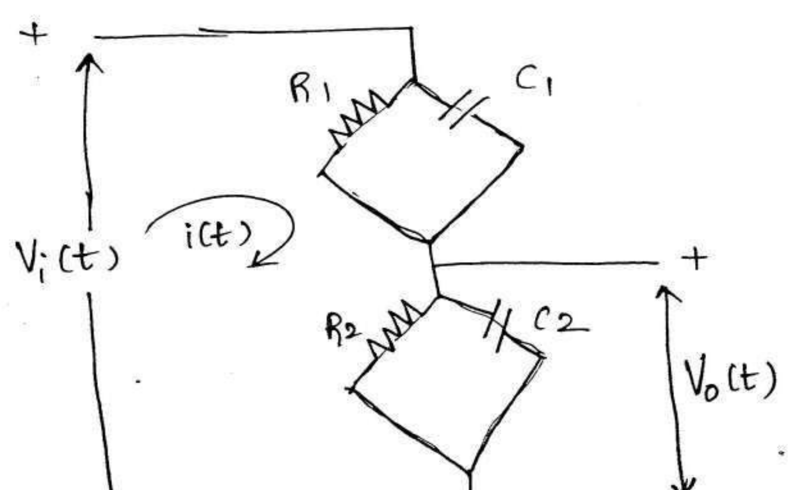
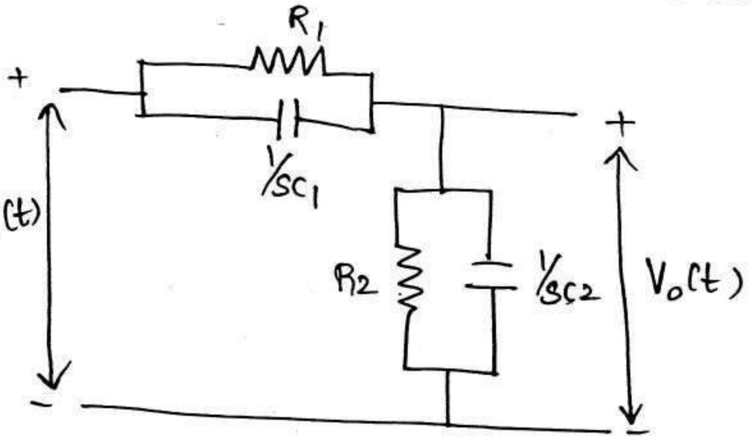
4. Obtain Transfer function for following electrical Network.



Solution:-

Step 1 :- Redrawn the circuit in terms of 's' domain.



step 2:-

Apply Voltage divider rule,

$$V_o(s) = V_i(s) \frac{\left(R_2 \,||\, \frac{1}{sC_2}\right)}{\left(R_1 \,||\, \frac{1}{sC_1}\right) + \left(R_2 \,||\, \frac{1}{sC_2}\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\dfrac{R_2/sC_2}{R_2 + \frac{1}{sC_2}}}{\left(\dfrac{R_1/sC_1}{R_1 + \frac{1}{sC_1}}\right) + \left(\dfrac{R_2/sC_2}{R_2 + 1/sC_2}\right)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 / R_2C_2s + 1}{\dfrac{R_1}{1+R_1C_1s} + \dfrac{R_2}{1+R_2C_2s}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 / 1+\cancel{R_2}C_2s}{\ldots} = R_2(1+R_1C_1s$$

$$\frac{V_o(S)}{V_i(S)} = \frac{R_2(R_1C_1S+1)}{R_1(1+R_2C_2S)+R_2(1+R_1C_1S)}$$

$\therefore$ Transfer function $G(S) = \frac{V_o(S)}{V_i(S)} = \frac{R_2 + R_1R_2C_1S}{R_1 + R_2 + SR_1R_2C_2 + SR_1R_2C_1}$ //.

$$V_i(S) \longrightarrow \boxed{\frac{R_2 + R_1R_2C_1S}{R_1 + R_2 + SR_1R_2C_2 + SR_1R_2C_1}} \longrightarrow V_o(S).$$

Block Diagram Representation.

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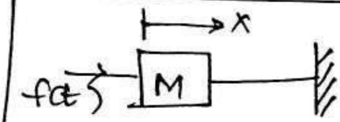
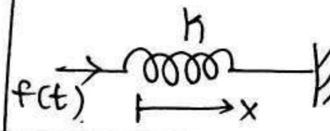
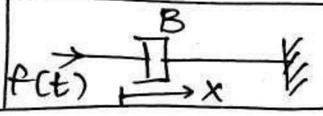
Modelling of Mechanical systems.

The equations of motions are generally formulated using Newtons law of motion. A Mechanical system may have either

Translational Motion	Rotational Motion
[Motion take place in straight lines]	[Motion will be in rotation]
I/p → Force (Newton)	I/P → Torque (N-m)
O/p → Displacement. (metre)	O/P → Angular Displacement (θ).
1. Mass (M) (kg)	1. Moment of Inertia (J)

Translational Mechanical system.

Sno	Element	symbol	Representation	Time domain	Laplace Domain
1	Mass	M		$f(t) = M\frac{d^2x}{dt^2}$	$F(S) = MS^2X(S)$
2	Spring	K		$f(t) = Kx$	$F(S) = KX(S)$
3	Damper	B		$f(t) = B\cdot\frac{dx}{dt}$	$F(S) = BSX(S)$

<u>Mass</u> stores Translational Kinetic Energy due to its Velocity. while <u>spring</u> stores Translational Potential Energy which is due to its position. <u>Damper</u> or Dashpot is capable of dissipating energy.

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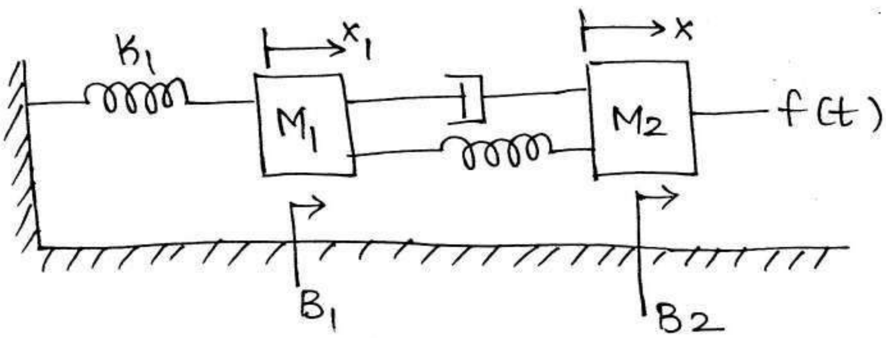
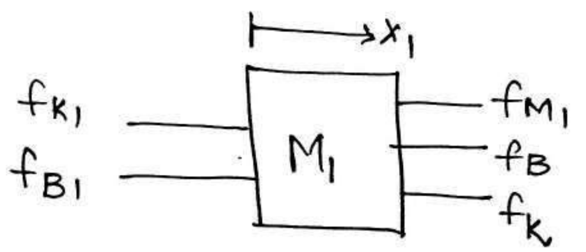
1. write the Differential Equation governing the translational Mechanical system as shown in Fig. & determine the Transfer function.



<u>Hint</u> :- $\boxed{T \cdot F = \frac{X(S)}{F(S)}}$

Solution:-

Step 1: Draw the free body diagram of Mass element 'M_1'.



By using Newtons II law of Motion,

Sum of opposing force = sum of Applying force.

$$f_{M1} + f_B + f_K + f_{K1} + f_{B1} = 0.$$

Differential equation:-

$$\boxed{M_1 \frac{d^2x_1}{dt^2} + B\frac{d}{dt}(x_1 - x) + K(x_1 - x) + K_1 x_1 + B_1 \frac{dx_1}{dt} = 0}$$

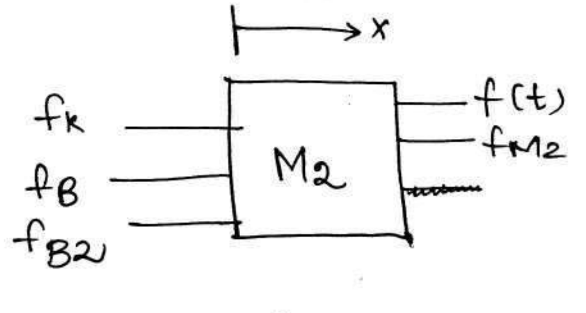
Step 2:- Taking Laplace Transform on both sides.

$$M_1 s^2 X_1(s) + Bs\left[X_1(s) - X(s)\right] + K\left[X_1(s) - X(s)\right] + K_1 X_1(s) + B_1 s\, X_1(s) = 0.$$

$$X_1(s)\left[M_1 s^2 + K_1 + B_1 s + Bs + K\right] - X(s)\left[Bs + K\right] = 0.$$

$$\boxed{X_1(s) = \frac{X(s)(Bs + K)}{M_1 s^2 + (K_1 + K) + (B_1 + B)s}}$$

Step 3:- Draw the free Body diagram of Mass element 'M_2'.



By, Newtons II law of motion:-

$$f(t) = f_{M2} + f_K + f_B + f_{B2}$$

$$\boxed{f(t) = M_2 \frac{d^2x}{dt^2} + K(x - x_1) + B\frac{d}{dt}(x - x_1) + B_2\frac{dx}{dt}}.$$

Step 4: Taking Laplace Transform on both sides,

$$F(s) = M_2 s^2 X(s) + K(X(s) - X_1(s)) + BS(X(s) - X_1(s)) + B_2 S X(s).$$

$$F(s) = [M_2 s^2 + (B_2 S + BS) + K]X(s) - [BS + K]X_1(s).$$

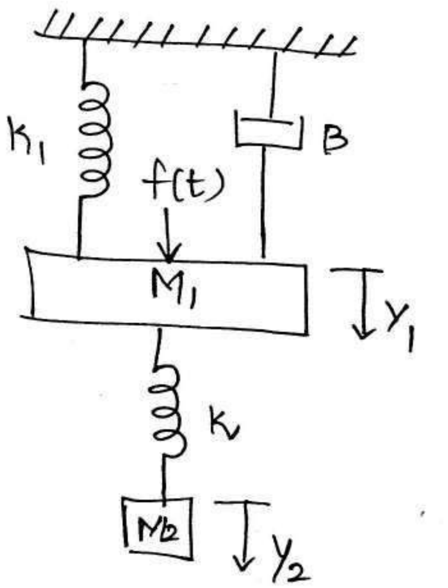
Substitute $X_1(s)$ Value to above equation.

$$F(s) = [M_2 s^2 + (B_2 S + BS) + K]X(s) - [BS + K]\left[\frac{X(s)(BS+K)}{M_1 s^2 + (K_1 + K) + (B_1 + B)S}\right]$$

$$F(s) = \left[(M_2 s^2 + (B_2 S + BS) + K) - \frac{(BS+K)^2}{M_1 s^2 (K_1 + K)(B_1 + B)S}\right] X(s)$$

$$\therefore T.F = G(s) = \frac{X(s)}{F(s)} = \frac{M_1 s^2 + (B_1 + B)S + (K_1 + K)}{[M_2 s^2 + (B2S + BS) + K)(M_1 s^2 (K_1 + K)(B_1 + B)S - (BS + K)^2]}$$

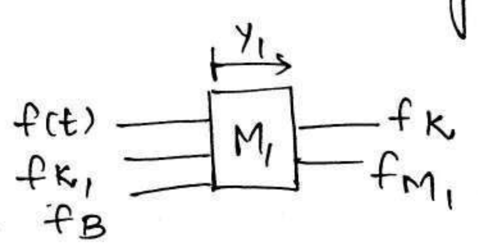
2. Determine Transfer function for given diagram.



(∵ Hint : $T.F = \frac{Y_2(s)}{F(s)}$)

⇒ Solution :-

Step 1 :- Draw Free body Diagram of mass 'M1' :-



By Newtons II law,

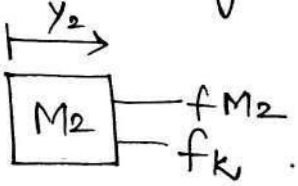
$$f(t) = f_{K1} + f_B + f_K + f_{M_1}$$

$$\boxed{f(t) = K_1 Y_1 + B\frac{dY_1}{dt} + K(Y_1 - Y_2) + M_1\frac{d^2Y_1}{dt^2}}$$

Step 2 :- Apply Laplace Transform on both sides,

$$F(s) = K_1 Y_1(s) + Bs Y_1(s) + K[Y_1(s) - Y_2(s)] + M_1 s^2 Y_1(s)$$

Step 3 :- Draw Free body Diagram for mass 'M_2':



By Newtons II law of Motion,

$$f_{M_2} + f_K = 0$$

$$M_2 \cdot \frac{d^2 y_2}{dt^2} + K(y_2 - y_1) = 0$$

Step 4 :- Apply Laplace transform,

$$M_2 s^2 Y_2(s) + K\,(Y_2(s) - Y_1(s)) = 0$$

$$[M_2 s^2 + K]\,Y_2(s) - K\,Y_1(s) = 0$$

$$(M_2 s^2 + K)\,Y_2(s) - K\left(\frac{F(s) + K\,Y_2(s)}{M_1 s^2 + Bs + (K_1 + K)}\right) = 0$$

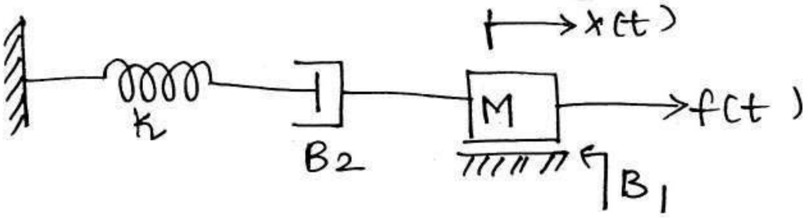
$$Y_2(s)\left[M_2 s^2 + K - \frac{K^2}{M_1 s^2 + Bs + K_1 + K}\right] = \frac{K\,F(s)}{M_1 s^2 + Bs + K_1 + K}$$

Transfer function :- (Take L.C.M & Denom. get cancel)

$$\boxed{G(s) = \frac{Y_2(s)}{F(s)} = \frac{K}{(M_2 s^2 + K)(M_1 s^2 + Bs + K_1 + K) - K^2}}$$

—— x —— x ——

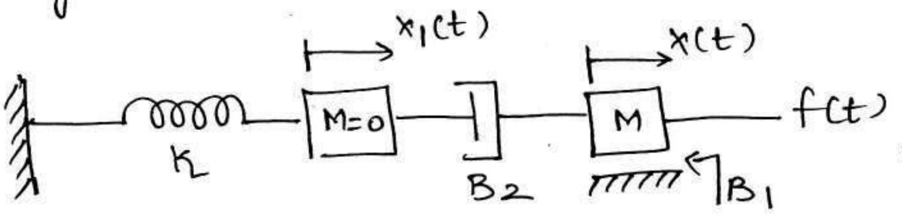
3. Determine Transfer function for the system shown in Figure,



= Solution:-

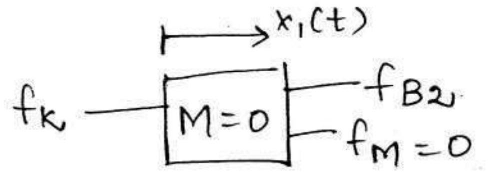
let x_1 be the displacement at meeting point of spring and dashpot. Hence system has two nodes, mass M & meeting point of spring and dashpot.

∴ Diagram becomes,



<u>Step 1</u> :-

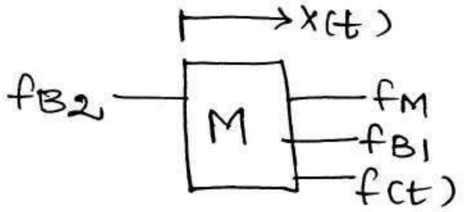
Free Body diagram of mass $\boxed{M=0}$.



$$0 = f_K + f_{B2}$$

$$0 = K\,x_1(t) + B_2 \frac{d}{dt}(x_1(t) - x(t))$$

<u>step 3</u>:- Draw Free body diagram of mass element M,



By Newtons II law of motion:-

$$f(t) = f_M + f_{B1} + f_{B2}.$$

$$f(t) = M\frac{d^2x}{dt^2} + B_1\frac{dx}{dt} + B_2\frac{d(x-x_1)}{dt}.$$

Taking Laplace Transform on both sides,

$$Ms^2X(s) + B_1 sX(s) + B_2 s[X(s) - X_1(s)] = F(s).$$

$$\left[Ms^2 + (B_1 + B_2)s\right]X(s) - B_2 s X_1(s) = F(s).$$

Substitute $X_1(s)$ value to above equation.

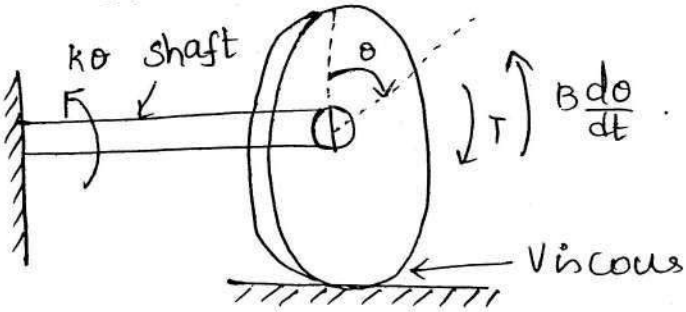
$$\left[Ms^2 + (B_1 + B_2)s\right]X(s) - B_2 s\left[\frac{B_2 s}{B_2 s + K}\right]X(s) = F(s).$$

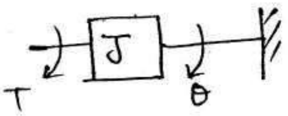
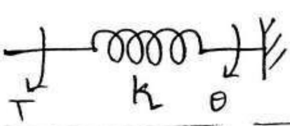
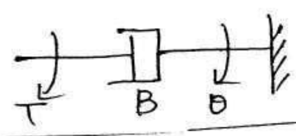
$$\boxed{T.F = \frac{X(s)}{F(s)} = \frac{B_2 s + K}{\left[Ms^2 + (B_1 + B_2)s\right](B_2 s + K) - (B_2 s)^2}}$$

//.

Rotational Mechanical System.

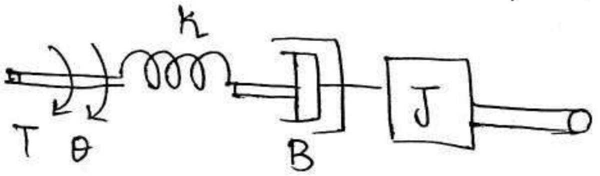
Mechanical systems involving rotation around a fixed axis are often seen in Machineries such as turbines, pumps, rotating discs, gears, generators, motor and so on. It consists of rotating disc of moment of inertia J, shaft stiffness K. The disc rotates in a viscous medium with a viscous friction coefficient B.



Sno	Element	Symbol	Time do Representation	Time domain	Laplace domain
1	Moment of inertia	J		$T = J\frac{d^2\theta}{dt^2}$	$T(S) = JS^2\theta(S)$
2	shaft stiffness	K		$T = K\theta$	$T(\theta) = K\theta(S)$
3	Viscous friction coefficient	B		$T = B\frac{d\theta}{dt}$	$T(S) = BS\theta(S)$

$$\boxed{\text{Applied Torque} = \text{Inertia Torque} + \text{Damping Torque} + \text{Torsional Torque}}$$

1.



Determine Transfer function for given Rotational mechanical system.

= Solution :-

$$T = T_k + T_B + T_J$$

$$T = k\theta + B\frac{d\theta}{dt} + J\frac{d^2\theta}{dt^2} \longrightarrow ①$$

Taking Laplace Transform on both sides,

$$T(s) = k\theta(s) + Bs\,\theta(s) + Js^2\theta(s)$$

$$T(s) = [k + Bs + Js^2]\,\theta(s) \longrightarrow ②$$

Transfer function : $G(s) = \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + k}$

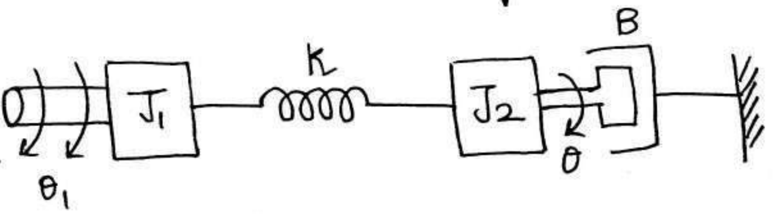
$$\boxed{G(s) = \frac{1}{Js^2 + Bs + k}}$$

Block Diagram Representation:-

$$T(s) \longrightarrow \boxed{G(s) = \frac{1}{Js^2 + Bs + k}} \xrightarrow{\theta(s)}$$

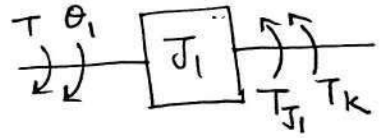
Rotational Mechanical system.

1. Determine Transfer function for the following Mechanical Rotational system.



Solution:

Step 1: Draw Free Body diagram of moment of inertia element 'J_1'.



By using Newtons II law of motion.

Sum of opposing torque = sum of Applying torque

$$T_{J_1} + T_K = T$$

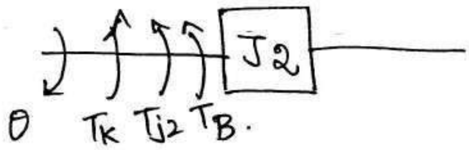
$$\boxed{J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T} \longrightarrow (1)$$

Step 2:- Taking Laplace Transform on bothsides,

$$J_1 s^2 \theta_1(s) + K[\theta_1(s) - \theta(s)] = T(s)$$

$$\theta_1(s)[J_1 s^2 + K] - K\theta(s) = T(s).$$

Step 3: Draw free body diagram of moment of inertia element J_2.



By using Newtons II law of motion,

$$T_{j2} + T_B + T_K = 0.$$

$$\boxed{J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0.} \longrightarrow (3)$$

step (4): Taking Laplace Transform on both sides.

$$J_2 S^2 \theta[S] + BS\,\theta(S) + K(\theta(S) - \theta_1(S)) = 0.$$

$$\boxed{\theta(S)\left[J_2 S^2 + BS + K\right] - K\,\theta_1(S) = 0.} \longrightarrow (4)$$

Substitute $\theta_1(S)$ Value in Eq (4).

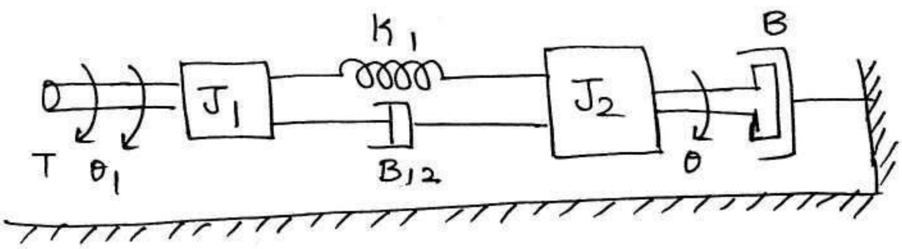
$$\theta(S)\left[J_2 S^2 + BS + K\right] - K\left[\frac{T(S) + K\,\theta(S)}{J_1 S^2 + K}\right] = 0.$$

$$\theta(S)\left[J_2 S^2 + BS + K\right] - \frac{K\,T(S)}{J_1 S^2 + K} - \frac{K^2\theta(S)}{J_1 S^2 + K} = 0$$

$$\theta(S)\left[J_2 S^2 + BS + K - \frac{K^2}{J_1 S^2 + K}\right] = \frac{K\,T(S)}{J_1 S^2 + K}.$$

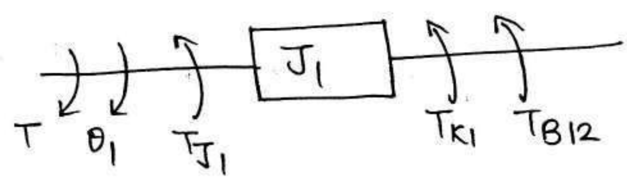
Step 5:- Transfer function : $\boxed{G(S) = \frac{\theta(S)}{T(S)}}$

2. write the Differential equation for the system shown in figure & determine Transfer function.



= Step 1 :-

Draw Free Body diagram of moment of inertia J_1



By using Newtons II law of motion,

$$T_{J_1} + T_{K_1} + T_{B12} = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K_1(\theta_1 - \theta) + B_{12}\frac{d}{dt}(\theta_1 - \theta) = T \longrightarrow (1)$$

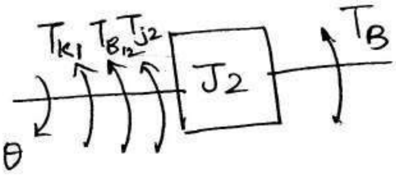
Step 2 :- Taking Laplace Transform on both sides,

$$J_1 s^2 \theta_1(s) + K_1[\theta_1(s) - \theta(s)] + B_{12}[\theta_1(s) - \theta(s)] = T(s).$$

$$\theta_1(s)[J_1 s^2 + K_1 + B_{12}s] - \theta(s)[K_1 + B_{12}s] = T(s).$$

$$\boxed{\theta_1(s) = \frac{T(s) + \theta(s)[K_1 + B_{12}s]}{J_1 s^2 + B_{12}s + K_1}} \longrightarrow (2)$$

Step 3:- Draw free Body diagram of moment of inertia 'J_2'.



By using Newtons II law of motion,

$$T_{J2} + T_B + T_{K1} + T_{B12} = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K_1(\theta - \theta_1) + B_{12}\frac{d}{dt}(\theta - \theta_1) = 0 \longrightarrow (3)$$

Step 4:- Taking Laplace Transform on both sides,

$$J_2 S^2 \theta(S) + BS\theta(S) + K_1[\theta(S) - \theta_1(S)] + B_{12}S[\theta(S) - \theta_1(S)] = 0$$

$$\theta(S)[J_2 S^2 + BS + K_1 + B_{12}S] - \theta_1(S)[K_1 + B_{12}S] = 0 \longrightarrow (4)$$

Substitute $\theta_1(S)$ from eq (2) in eq (4).

$$\theta(S)[J_2 S^2 + BS + K_1 + B_{12}S] - \left[\frac{T(S) + \theta(S)[K_1 + B_{12}S]}{J_1 S^2 + B_{12}S + K_1}\right](K_1 + B_{12}S) = 0$$

$$\theta(S)\left[\frac{[J_1 S^2 + B_{12}S + K][J_2 S^2 + BS + B_{12}S + K_1] - [K_1 + B_{12}S]^2}{[J_1 S^2 + B_{12}S + K_1]}\right] = \frac{T(S)[K_1 + B_{12}S]}{J_1 S^2 + B_{12}S + K_1}$$

Transfer function :-
$$\boxed{\frac{\theta(S)}{T(S)} = \frac{K_1 + B_{12}S}{[J_1 S^2 + B_{12}S + K][J_2 S^2 + BS + B_{12}S + K_1] - \ldots}}$$

ANALOGIOUS SYSTEM
(similar)

Sometimes Mechanical and other systems are converted into electrical analogous systems for the easy q design, modification and analysis. Analogous systems have same type q differential equations. There are four types q Analogies namely,

(i) Force - Voltage Analogy
(ii) Force - current Analogy
(iii) Torque - Voltage Analogy
(iv) Torque - current Analogy.

The Translational Mechanical system is represented by mass, spring and damper in equation,

$$\boxed{M\frac{d^2x}{dt^2} + B\cdot\frac{dx}{dt} + Kx = f(t)} \longrightarrow (1)$$

The Rotational Mechanical system is represented by moment q inertia, ~~spring~~ shaft and viscous friction coefficient in equation,

$$\boxed{J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T} \longrightarrow (2)$$

The Electrical system is represented by Inductor, Resistor, capacitor in terms q voltage

$$V(t) = L\frac{di(t)}{dt} + R\,i(t) + \frac{1}{C}\int i\,dt.$$

But $\Rightarrow$ $\boxed{i(t) = \frac{dq}{dt}}$ ie, current is rate of flow of charge.

$$\therefore \boxed{V(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q} \longrightarrow (3)$$

The electrical system can be represented by capacitor, Resistor, inductor in terms of current.

$$\boxed{i(t) = \frac{1}{L}\int e\,dt + C\frac{de}{dt} + \frac{1}{R}e}$$

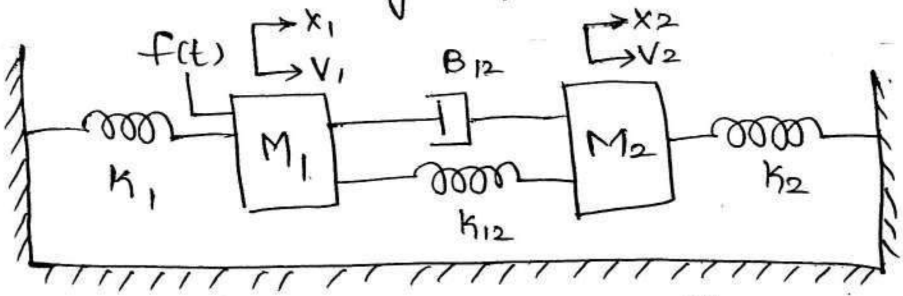
But $\boxed{e = \frac{d\psi}{dt}}$ where $\psi \rightarrow$ flux linkage.

$$\boxed{i(t) = C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi} \longrightarrow (4)$$

compare equation (1), (2), (3), (4) which gives equivalent parameters. It is given by Table.

Sno	Translational	Rotational	Electrical s/m in terms of Voltage	Electrical s/m in terms of I
1	F	T	$V(t)$	$i(t)$
2	M	J	L	C
3	B	B	R	$1/R$
4	K	K	$1/C$	$1/L$

1. Draw the Force-Voltage & Force current analogy for the following system as shown in Fig.

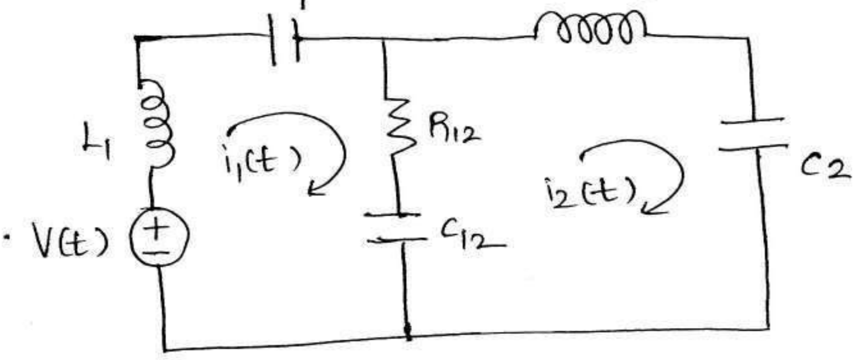


Step 1 :-

Identify equivalent electrical systems in terms of voltage.

$f(t) = V(t)$	$B_{12} = R_{12}$.	$K_2 = 1/C_2$
$K_1 = 1/C_1$	$K_{12} = 1/C_{12}$	
$M_1 = L_1$	$M_2 = L_2$	
$V_1 \rightarrow i_1(t)$	$V_2 = i_2(t)$	

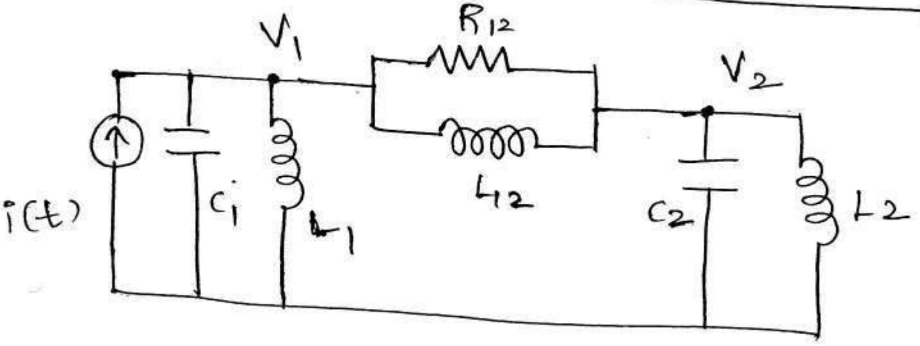
Step 2 :- Force - Voltage circuit diagram :-



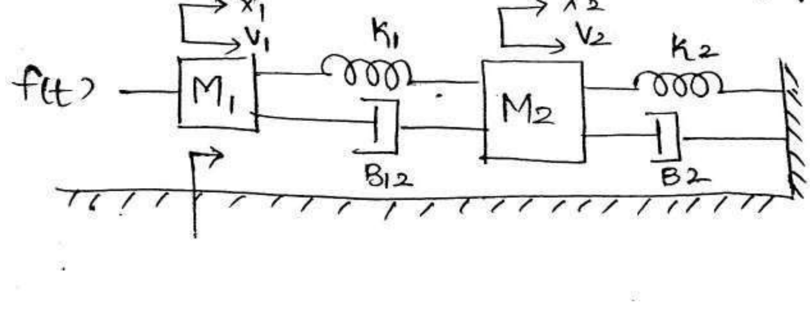
Step 3: Identify equivalent electrical terms in terms of current.

$f(t) = i(t)$	$K_1 = 1/L_1$
$V_1 = V_1(t)$	$K_2 = 1/L_2$
$V_2 = V_2(t)$	$K_{12} = 1/L_{12}$
$M_1 = C_1$	$B_{12} = 1/R_{12}$
$M_2 = C_2$	

Step 4 :- Force - Current circuit Diagram.

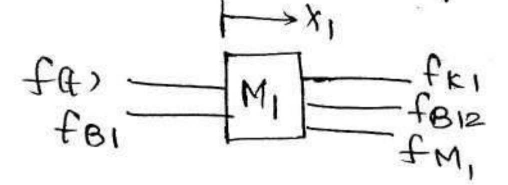


2. write a differential equation for mechanical system as shown in Figure. Draw Force - Voltage & Force - current electrical analogy circuit.



Step 1 :-

Draw free body diagram,



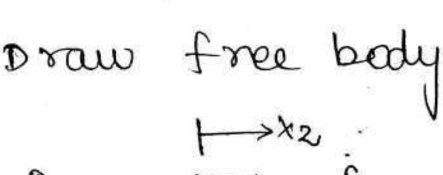
By using Newtons II law,

$$f(t) = f_{M_1} + f_{K_1} + f_{B12} + f_{B1}$$

$$\boxed{f(t) = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 (x_1 - x_2) + B_{12} \frac{d}{dt}(x_1 - x_2)} \rightarrow (1)$$

Step 2 :

Draw free body diagram,



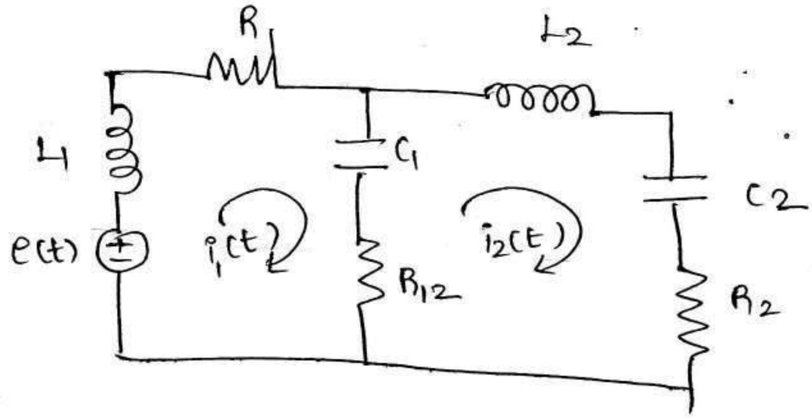
By Newtons II law,

$$f_{M_2} + f_{K_2} + f_{B_2} + f_{K_1} + f_{B12} = 0 .$$

$$\boxed{M_2 \frac{d^2 x_2}{dt^2} + K_2 x_2 + B_2 \frac{dx_2}{dt} + K_1 (x_2 - x_1) + B_{12} \frac{d}{dt}(x_2 - x_1) = 0} \rightarrow (2)$$

Step 3: Force - Voltage Analogy.

$f(t) = e(t)$ $\quad M_1 = L_1$ $\quad B_1 = R_1$ $\quad K_1 = 1/C_1$



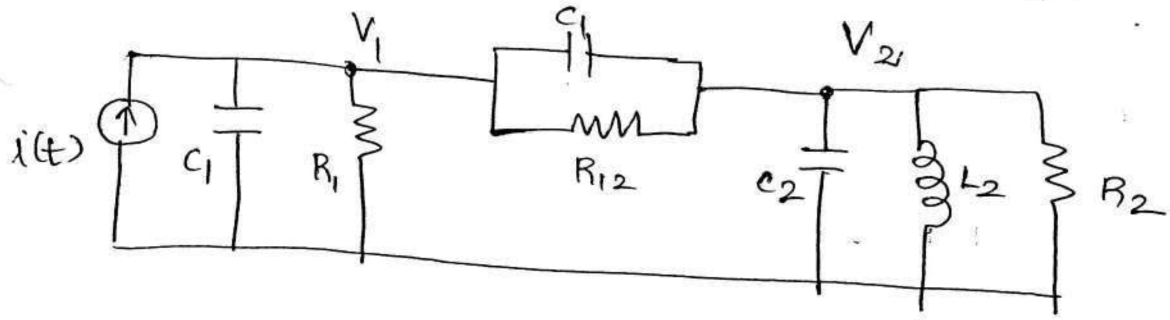
Step 4:- Force current Analogy system.

$f(t) = \hat{i}(t)$ $\quad M_1 = C_1$ $\quad K_1 = 1/L_1$ $\quad B_1 = 1/R_1$

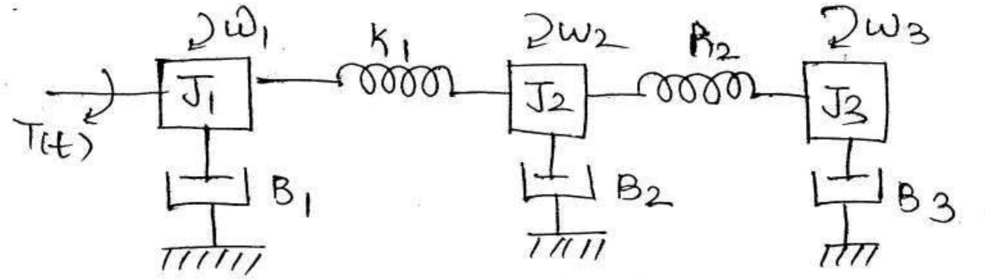
$V_1 = V_1(t)$ $\quad M_2 = C_2$ $\quad K_2 = 1/L_2$ $\quad B_2 = 1/R_2$

$V_2 = V_2(t)$

$B_{12} = 1/R_{12}$.



3. Draw the Torque-Voltage & Torque current Analogous system for the mechanical system.



Step 1:-

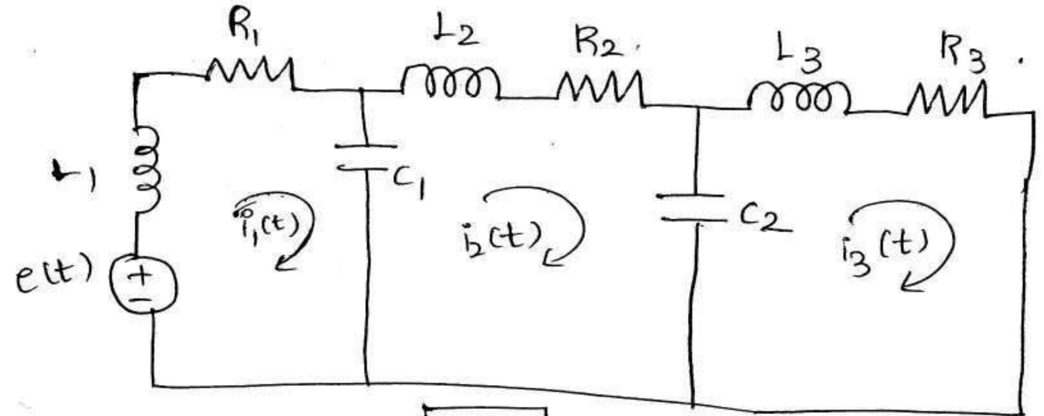
Identify equivalent electrical systems in terms of Voltage.

$T(t) = e(t)$ $\quad J_1 = L_1$ $\quad K_1 = 1/C_1$ $\quad B_1 = R_1$

$\omega_1 = i_1(t)$ $\quad J_2 = L_2$ $\quad K_2 = 1/C_2$ $\quad B_2 = R_2$

$\omega_2 = i_2(t)$ $\quad J_3 = L_3$ $\quad B_3 = R_3$

$\omega_3 = i_3(t)$



T-V Analogous S/m

Step 2:-

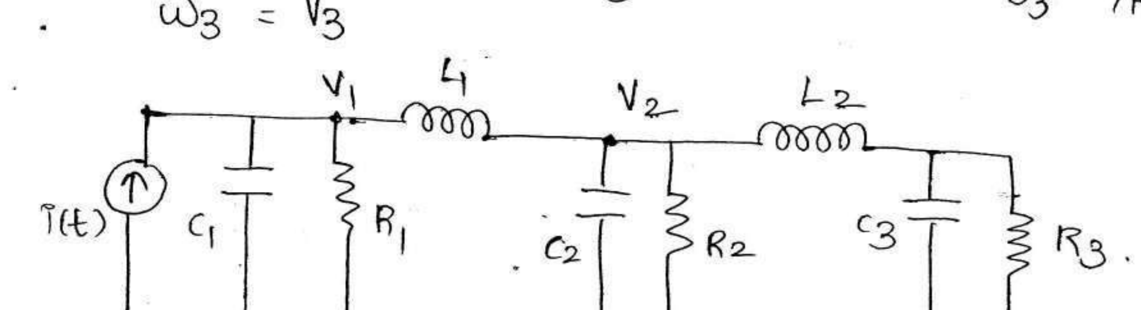
Identify equivalent electrical system in terms of current.

$T(t) = i(t)$ $\quad J_1 = C_1$ $\quad K_1 = 1/L_1$ $\quad B_1 = 1/R_1$

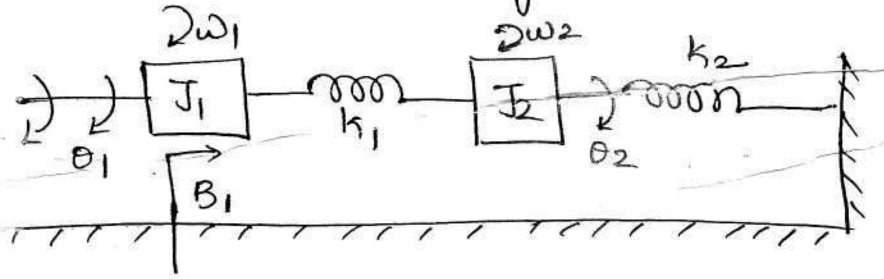
$\omega_1 = V_1$ $\quad J_2 = C_2$ $\quad K_2 = 1/L_2$ $\quad B_2 = 1/R_2$

$\omega_2 = V_2$ $\quad J_3 = C_3$ $\quad B_3 = 1/R_3$

$\omega_3 = V_3$

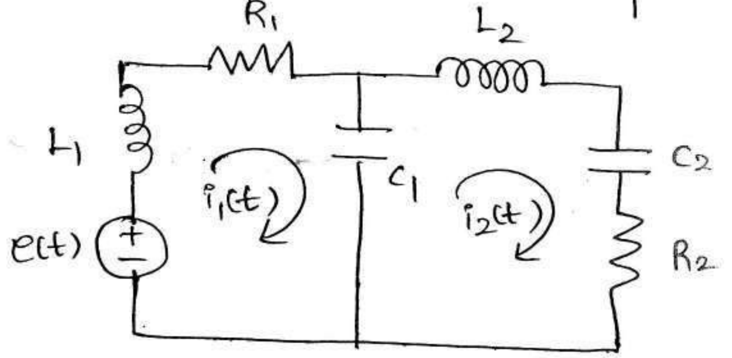


2. Draw the torque Voltage ~~torque~~ torque current analogy System for ~~the following~~ mechanical rotational s/m.



Step 1:- Draw Torque - Voltage circuit diagram.

$T = e(t)$	$J_1 = L_1$	$K_1 = 1/C_1$	$B_1 = R_1$
$\omega_1 = i_1(t)$	$J_2 = L_2$	$K_2 = 1/C_2$	$B_2 = R_2$
$\omega_2 = i_2(t)$			

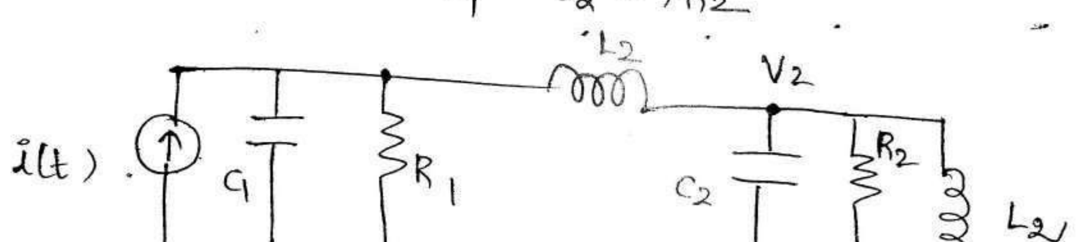


Step 2:- Draw Torque - current circuit diagram.

$T = i(t)$ $\quad J_1 = C_1$ $\quad K_2 = 1/L_2$

$\omega_1 = V_1(t)$ $\quad J_2 = C_2$ $\quad B_1 = 1/R_1$

$\omega_2 = V_2(t)$ $\quad K_1 = 1/L_1$ $\quad B_2 = 1/R_2$



BLOCK DIAGRAM REDUCTION TECHNIQUE

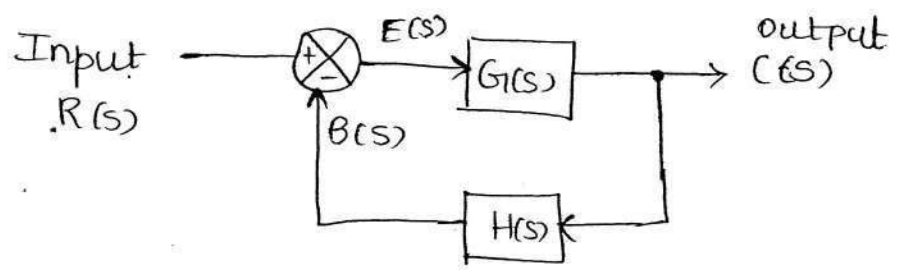
Any control system will have a number of control components. A control system can be represented in Block diagram form. The arrow head pointing towards a particular block indicates input to the system component and arrow head leading away from the block indicates output.

Block Diagram is possible to evaluate the contribution of each of components towards overall performance of control system.

It helps in understanding functional operation of the system more readily than examination of actual control system physically.

It may be noted that Block diagram drawn for a system is not unique.

Block diagram representation of closed loop:-



Block diagram is also called canonical form.

$$E(S) = R(S) - B(S)$$

$$B(S) = H(S).C(S)$$

$$\boxed{C(S) = E(S)G(S)}$$

$$R(S) = E(S) + B(S)$$

$$= E(S) + H(S)C(S)$$

$$= E(S) + H(S)E(S)G(S).$$

$$\therefore \boxed{R(S) = E(S) + H(S)E(S)G(S)}$$

$\therefore$ closed loop Transfer function $\Big\}$ $\dfrac{C(S)}{R(S)} = \dfrac{E(S)G(S)}{E(S) + H(S)E(S)G(S)}$

($\div$ by $E(S)G(S)$ on Nr & Dr) $= \dfrac{1}{\dfrac{1}{G(S)} + H(S)}$.

$T.F = \dfrac{C(S)}{R(S)} = \dfrac{G(S)}{1 + G(S)H(S)}$ $\longrightarrow$ Negative feedback

$T.F = \dfrac{C(S)}{R(S)} = \dfrac{G(S)}{1 - G(S)H(S)}$ $\longrightarrow$ positive feedback.

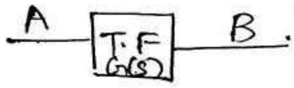
$T.F = \dfrac{C(S)}{R(S)} = \dfrac{G(S)}{1 \pm G(S)}$ $\longrightarrow$ unity feedback (ie., $H(S) = 1$)

——— x ———

Rules for Block Diagram simplification

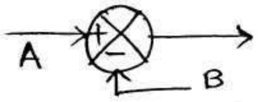
There are some rules which helps to simplify a block diagram of control system. and there are three basic elements in Block diagram.

(i) Block



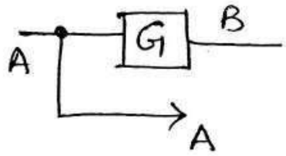
$\boxed{B = AG(s)}$

(ii) Summing point



$\boxed{A - B}$

(iii) Branching point



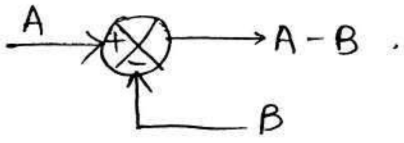
$\boxed{B = AG}$

<u>Block</u>: It is a symbol for mathematical operation on the input signal to the block that produces the output.

$\boxed{B = AG(s)}$

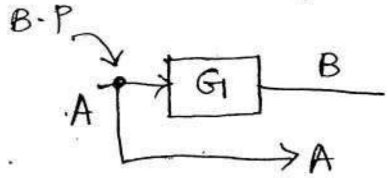
<u>Summing point</u>:

It is used to add two or more signals in the system.



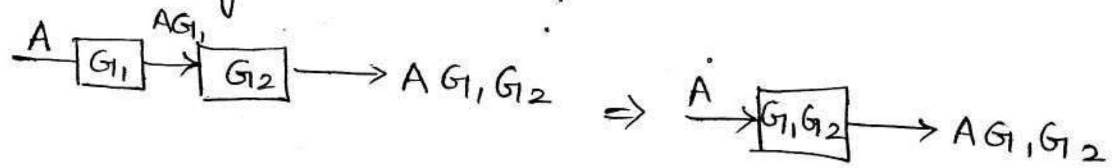
<u>Branch point</u>:

It is a point from which the signal from a block goes concurrently to other blocks or summing points.

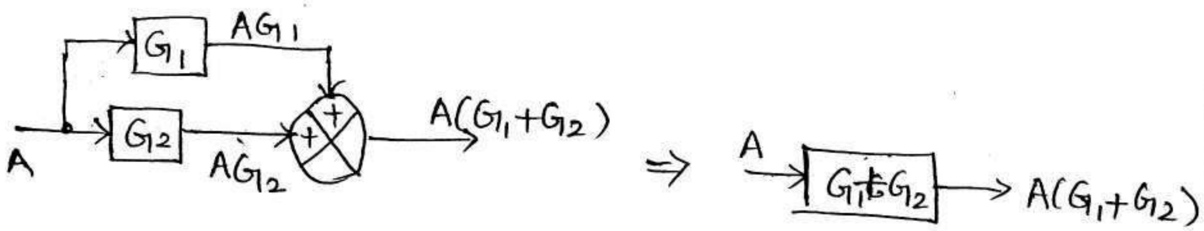


$\boxed{B = AG}$

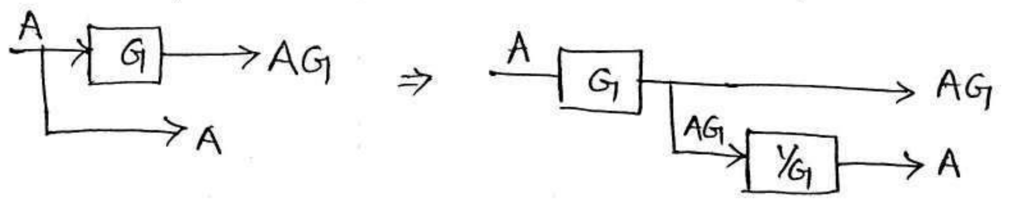
Rule 1: Combining the blocks in cascade



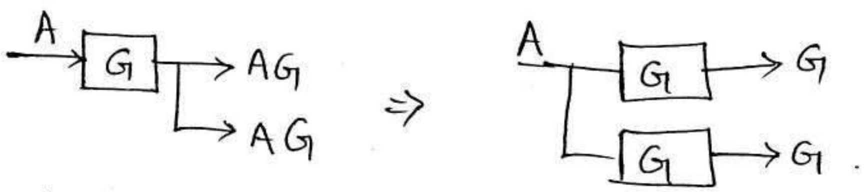
Rule 2: combining parallel blocks (or combining feed forward paths).



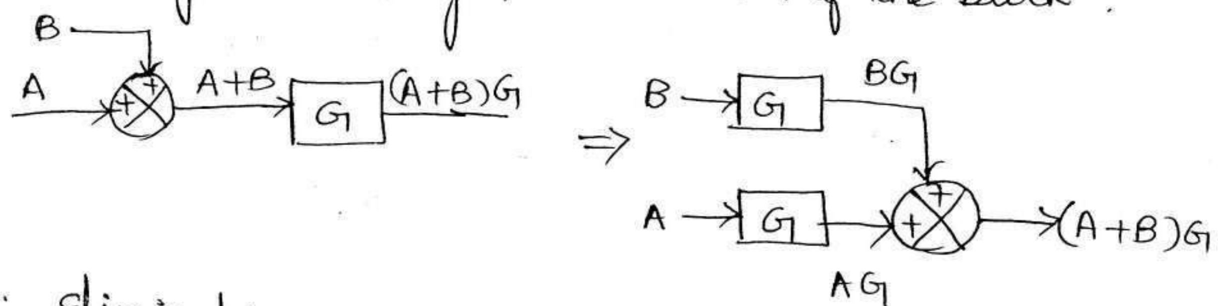
Rule 3: Moving the branch point ahead of block.



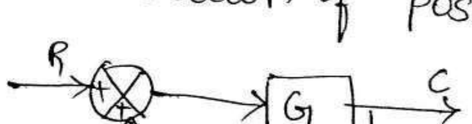
Rule 4: Moving branch point before the block.



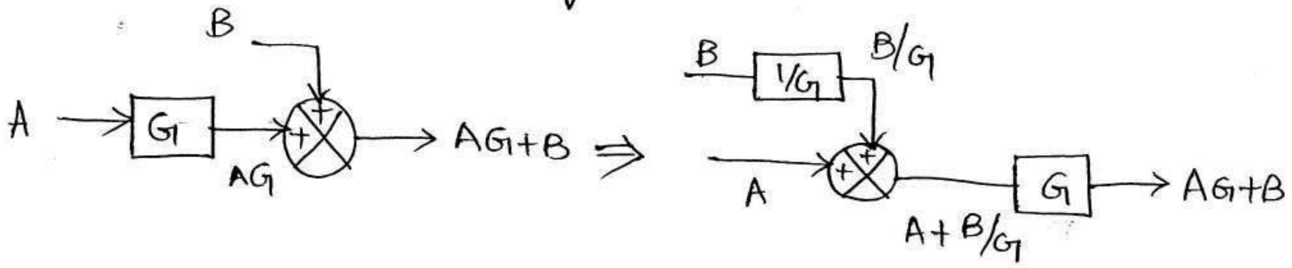
Rule 5: Moving summing point ahead of the block.



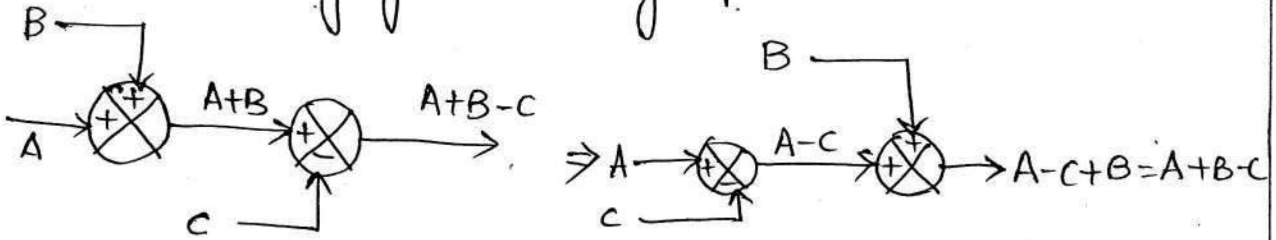
Rule 6: Elimination of positive feedback loop.



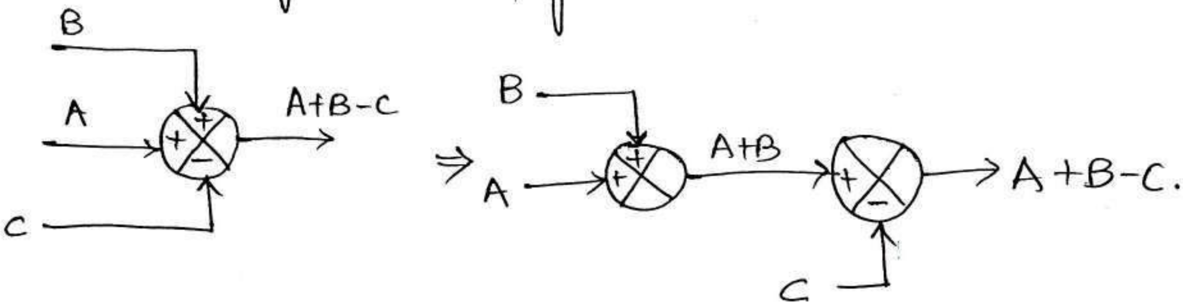
Rule 7:- Moving summing point before the block.



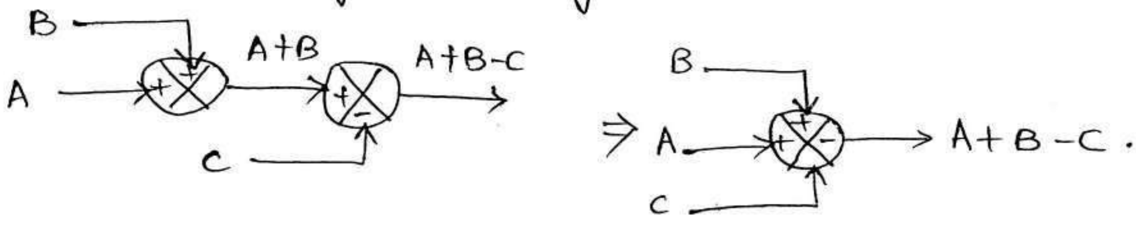
Rule 8: Interchanging summing point.



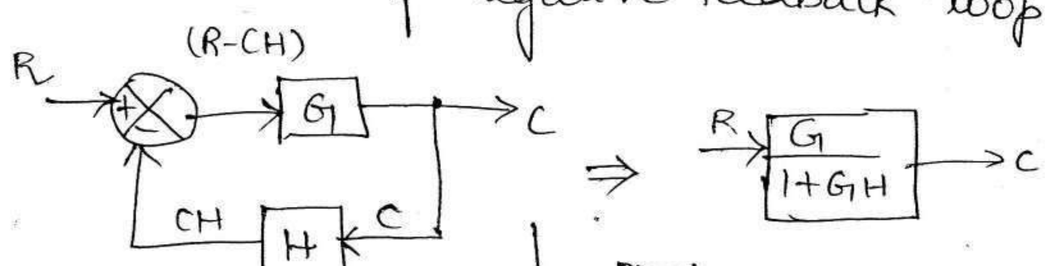
Rule 9: splitting summing points.



Rule 10: Combining summing Blockpoints.



Rule 11: Elimination of negative feedback loop.



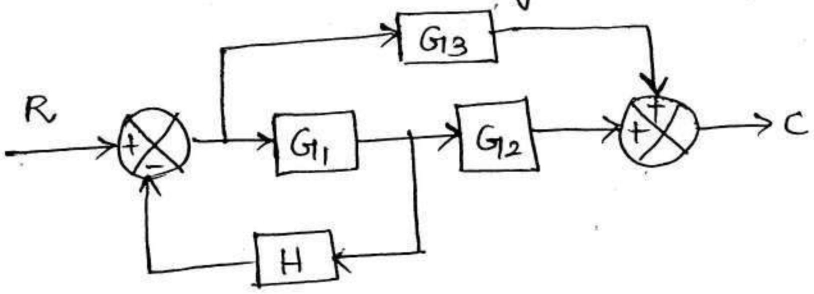
Note : * Branch points & summing points cannot be interchanged.

* No square terms in the Transfer function.

—— × —— × ——

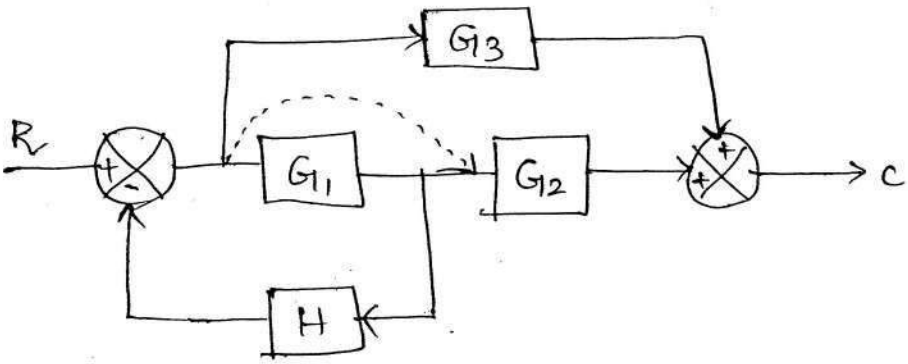
Problems

1. Reduce the block diagram and find C/R.



Solution:

step 1: Move the branch point after the block,



SIGNAL FLOW GRAPH MODEL:

For complicated systems, Block diagram reduction approach for arriving at transfer function relating the input and output Variables is tedious and time consuming.

Signal Flow graph (SFG) is an alternative approach developed by "S.J. Mason". It does not require any reduction process because of availability of flow graph gain formula which relates input and output System Variables.

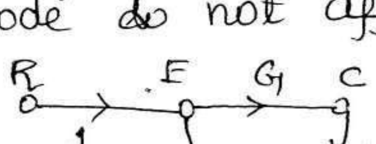
Definition:

A SFG is a graphical representation of relationship between the Variables of set of linear algebraic equations. It consists of a network in which nodes representing each of system variables are connected by directed branches.

Some important terms in SFG are as follows,

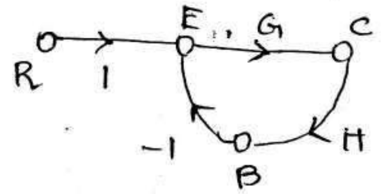
i) Node:

It represents a system Variable which is equal to sum of all incoming signals at the node. outgoing signals from node do not affect the Value of node Variable.



ii) <u>Branch</u> :-

A signal travels along a branch from one node to another in direction indicated by the branch arrow and in process, gets multiplied by gain or transmittance.

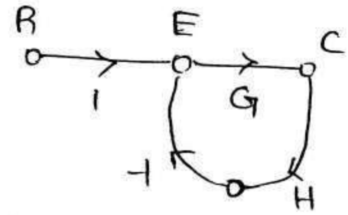


⇒ here, G is branch &

$\boxed{C = GE}$

iii) <u>Input node or source</u> :-

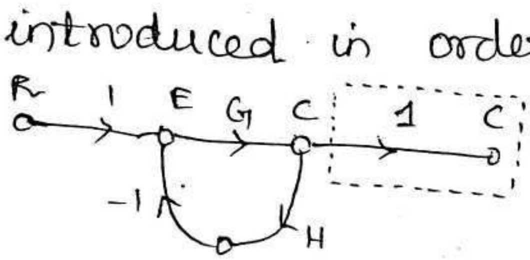
It is a node with only outgoing branches.



⇒ here R is input node or source.

(iv) <u>output node or sink</u>:

It is a node with only incoming branches. However this condition is not always met. An additional branch with unity gain may be introduced in order to meet the specified condition.



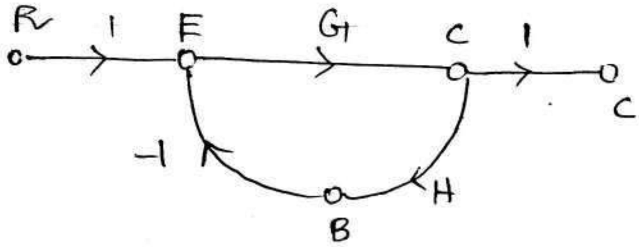
⇒ here C is output node or sink.

(v) <u>path</u>:

It is the traversal of connected branches in the direction of branch arrows such that no node is traversed more than once.

vi) <u>Forward path:</u>

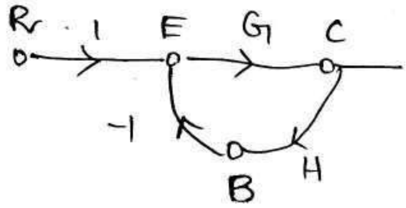
It is a path from input node to output node when no node encountered twice.



⇒ here R-E-C is a forward path.

vii) <u>Forward path gain:-</u>

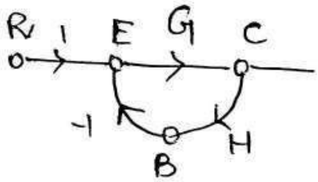
It is a product of branch gains in forward path.



⇒ here 'G' is forward path gain.

viii) <u>Loop:</u>

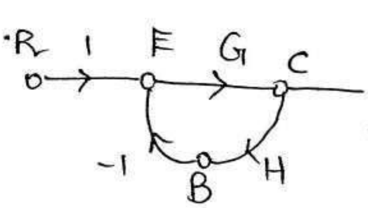
It is a path which originates and terminates at the same node.



⇒ here ECBE is a loop.

ix) <u>Loop gain:-</u>

It is a product of branch gains encountered in traversing the loop.



⇒ here "-GH" is loop gain for the loop E-C-B-E.

x) <u>Non-touching loop:-</u>

Mason's Gain Formula :-

According to Mason's gain formula, the overall gain 'T' is expressed as,

$$\boxed{T = \frac{1}{\Delta} \sum_{K=1}^{N} P_K \Delta_K .} = \frac{X_{out}}{X_{in}} .$$

where $P_K \rightarrow$ path gain of K^{th} forward path.

Δ = determinant of the path.

= 1 - (sum of loop gains of all individual loop) + (sum of gain products of all possible combinations of two non touching loops) - (sum of gain products of all possible combinations of three non touching loops) + ...

$$\boxed{\Delta = 1 - \sum_m P_{m1} + \sum_m P_{m2} - \sum_m P_{m3} + \cdots .}$$

Δ_K = Value of Δ for that part of graph non touching the K^{th} forward path.

T = overall gain of system.

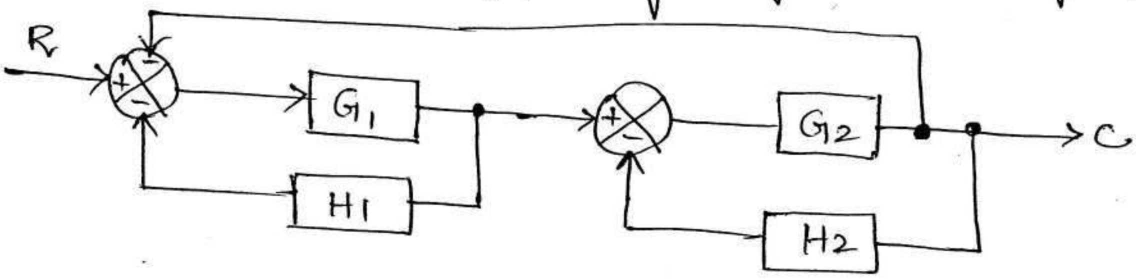
P_{mr} = gain product of m^{th} possible combination of 'r' non touching loops.

N = total number of forward paths.

— x — x —

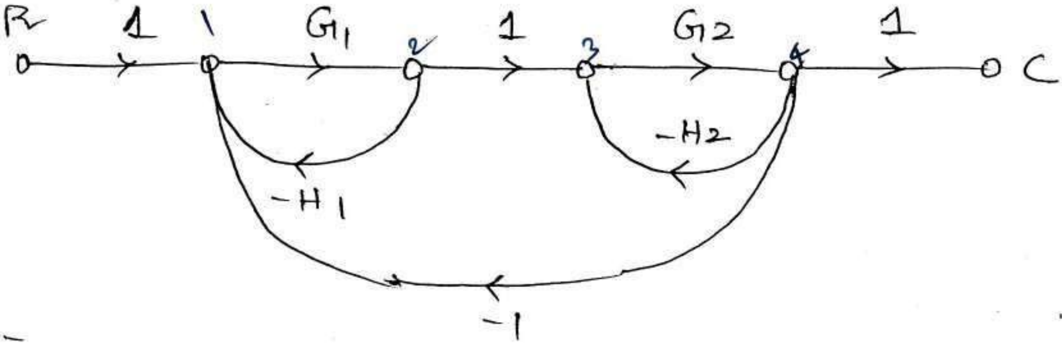
SFG problems are given by either equations or by Block diagrams or given as graph.

1. obtain the transfer function of a system given in Block diagram by using signal flow graph.



= Step 1:

The signal Flow graph for the system is drawn below,



Step 2:-

Finding out forward path.

here, Forward path is only one. i.e., R – C & its gain $= G_1G_2$

Step 3:

Finding out individual loops.

here, there are three individual loops, gains are

(i) $P_{11} = -G_1H_1$

(ii) $P_{12} = -H_2G_2$

(iii) $P_{13} = -G_1G_2$.

Step 4:

Finding out two non touching loops with gain.

$$P_{12} = (-G_1H_1)(-G_2H_2) = G_1G_2H_1H_2.$$

Step 5:

There is one forward path and all loops touch the forward path $\therefore \boxed{\Delta_1 = 1}$

Step 6: Finding out 'Δ'.

$\Delta = 1 -$ (Sum of loop gain of all indi. loops) + (Sum of gain product of all possible combination of two non touching) $- \ldots$

$$\boxed{\Delta = 1 - (-G_1H_1 - G_2H_2 - G_1G_2) + G_1H_1G_2H_2}$$

Step 7:- Mason's gain formula: $\boxed{T = \frac{1}{\Delta} \sum_K P_K \Delta_K}$

$$T = \frac{P_1\Delta_1}{\Delta} = \frac{G_1G_2}{1 + G_1H_1 + G_2H_2 + G_1G_2 + G_1G_2H_1H_2}$$

$(\because k=1)$

$$\boxed{T = \frac{G_1G_2}{(1+G_1H_1)(1+G_2H_2) + G_1G_2}}$$

— x — x —

2. Obtain the transfer functions of the following set of linear equation by using signal flow graph.

$$x_2 = t_{12}x_1 + t_{32}x_3$$
$$x_3 = t_{23}x_2 + t_{43}x_4$$
$$x_4 = t_{24}x_2 + t_{34}x_3 + t_{44}x_4$$
$$x_5 = t_{25}x_2 + t_{45}x_4.$$

Solution:

Here input node is x_1 and output node is x_5.

Step 1:-

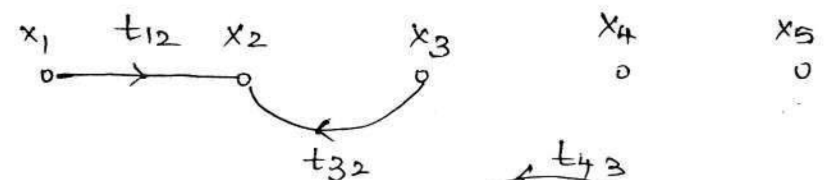
locate the nodes of the system. Totally 5 nodes. and it can be represented by,

x_1 x_2 x_3 x_4 x_5

Step 2:-

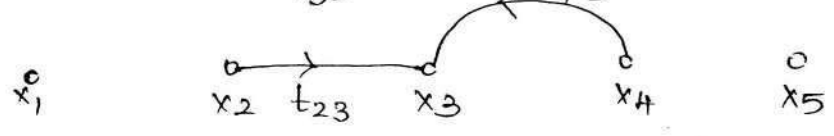
Draw SFG for four equations:

(i)



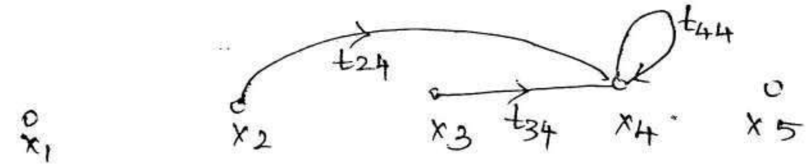
→ eq (1)

(ii)



→ eq (2)

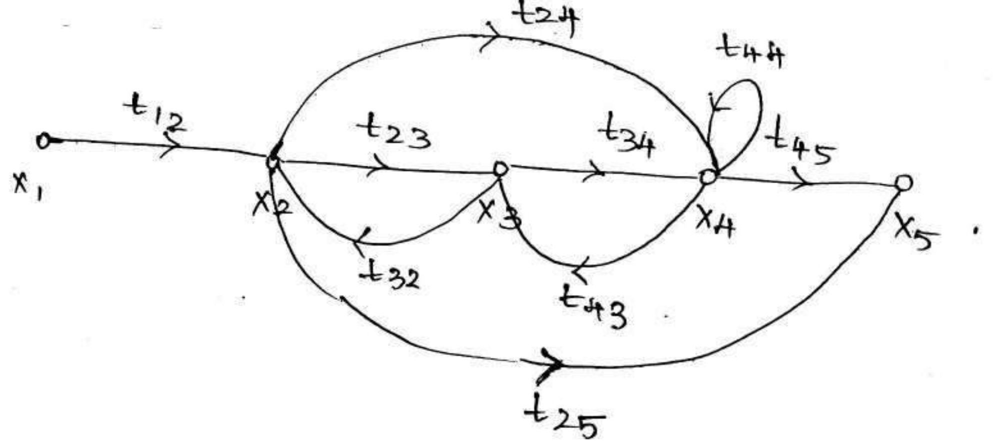
(iii)



→ eq (3)

Step 3 :-

over all signal flow graph is obtained by adding the graphs of individual equations.



Step 4 :- Finding out forward path gain.

$P_1 = t_{12}\, t_{23}\, t_{34}\, t_{45}$

$P_2 = t_{12}\, t_{24}\, t_{45}$

$P_3 = t_{12}\, t_{25}$.

Step 5 : Finding out individual feedback loops.

$P_{11} = t_{23}\, t_{32}$

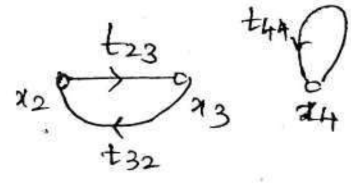
$P_{21} = t_{34}\, t_{43}$

$P_{31} = t_{44}$

$P_{41} = t_{32}\, t_{43}\, t_{24}$

Step 6 : There is only one possible combination of two non touching loops.

$P_{12} = t_{23}\, t_{32}\, t_{44}$



Step 7:-

The first forward path touches all loops, so no individual loop is formed $\therefore \boxed{\Delta_1 = 1}$

The second Forward path is eliminated, no individual loop is formed. $\therefore \boxed{\Delta_2 = 2}$

The third Forward path is eliminated, two loops are formed.

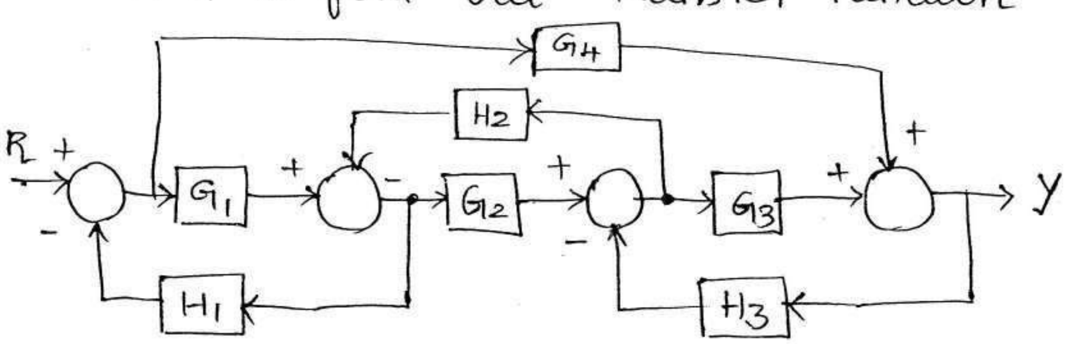
$$\therefore \boxed{\Delta_3 = 1 - t_{34}t_{43} - t_{44}}$$

Step 8:- Mason's gain Formula:-

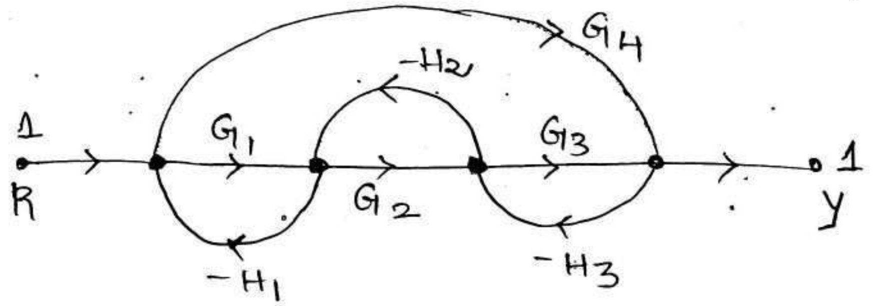
$$T.F = \frac{x_5}{x_1} = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3}{\Delta}$$

$$T.F = \frac{(t_{12}t_{23}t_{34}t_{45}) + (t_{12}t_{24}t_{45}) + t_{12}t_{25}(1 - t_{34}t_{43} - t_{44})}{1 - (t_{23}t_{32} + t_{34}t_{43} + t_{44} + t_{32}t_{43}t_{24}) + t_{23}t_{32}t_{44}} \quad //.$$

3) consider the block diagram as shown in Figure. Draw SFG & find out Transfer function.



Solution:- Negative feedback on block diagram are



⇒ Step ①

<u>Step 2</u>:- Finding out forward path with path gain.

$$\boxed{P_1 = G_1 G_2 G_3 \; ; \quad P_2 = G_4}$$

<u>Step 3</u>:- Finding out individual loops with loop gain

$P_{11} = -G_1 H_1$; $P_{21} = -H_2 G_2$; $P_{31} = -G_3 H_3$ $P_{41} = -G_4 H_3 H_2 H_1$

<u>step 4</u>: Finding out loop gain product of two non touching loops.

$\boxed{P_{12} = G_1 H_1 G_3 H_3}$ & there is no combination of more than two non touching loops.

$$\Delta = 1 - (-G_1 H_1 - G_2 H_2 - G_3 H_3 - G_4 H_3 H_2 H_1) + G_1 H_1 G_3 H_3$$

<u>Step 5</u>:- First forward path touches all loops

$\therefore \boxed{\Delta_1 = 1}$

Second forward path is not in touch with one loop. $\therefore \boxed{\Delta_2 = 1 - (-G_2 H_2)}$

<u>step 6</u>:- <u>Mason's Gain Formula</u>: $T = \frac{1}{\Delta} \sum_K P_K \Delta_K$

$= \frac{1}{\Delta} [P_1 \Delta_1 + P_2 \Delta_2]$

DC & AC Servomotor

The word servomechanism originated from the words "servant (or slave) & Mechanism". The motors that are used in automatic control system are called "servomotors". when the objective of the system is to control the position of an object then the system is called "servomechanism".

electrical signal to mechanical signal — **Servomotor** — converts an electrical signal to angular velocity or movement of shaft

Depends on supply req. to run motor:

- DC servomotor
  - Sliding contact motor
    - permanent Magnet motor
      - cylindrical armature motors
      - DISC armature motors
      - Moving coil motors
    - Electromagnetic field motor
      - Armature control motor
      - Field controlled motor
      - series motor
      - splitfield motor.
  - contact less motor.
- AC Servomotor

Requirements of Good servomotor:-

(i) It should be easily reversible, have linear torque-speed characteristics.

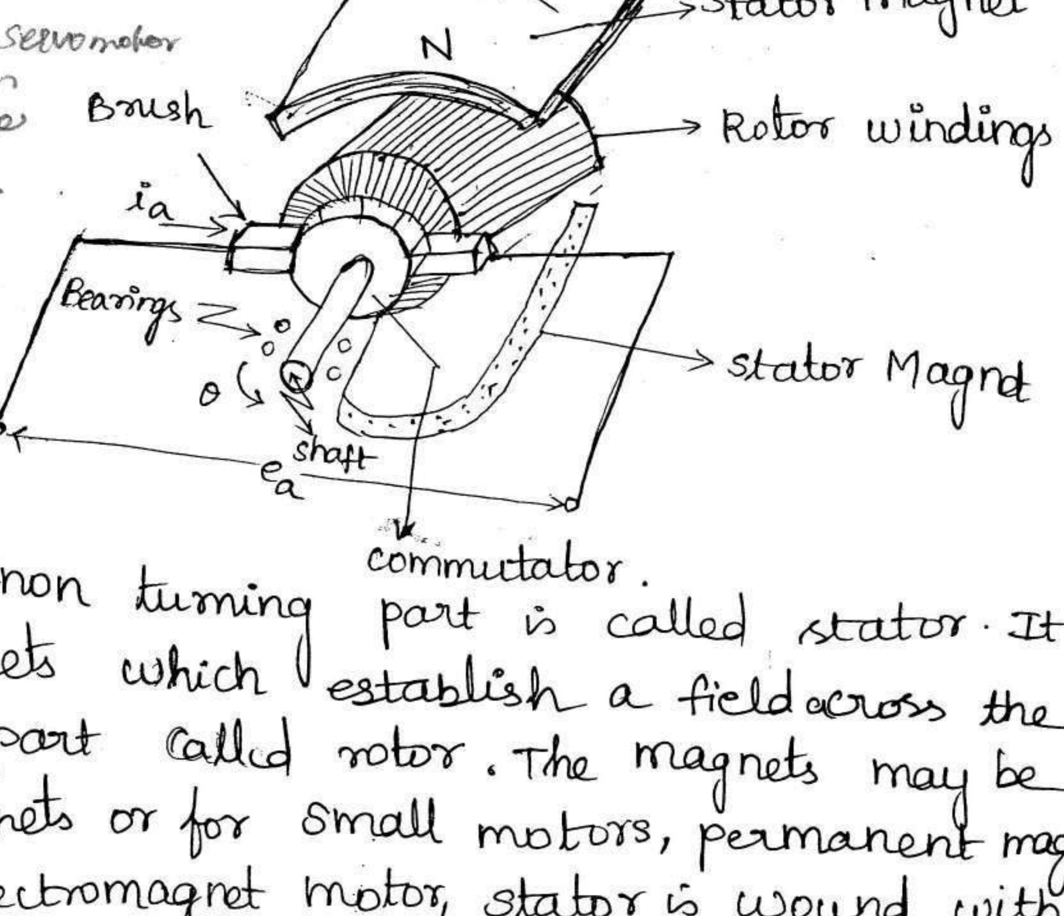
(iii) Linear relationship between speed and electric control signal.

(iv) low mechanical and electrical inertia.

(v) Fast response.

DC Servo motor :- → Same as DC motor.
→ Act as mechanical transducer which converts DC voltage into mechanical signal.

→ control of DC servomotor can be from
* Armature side
* Field side



The non turning part is called stator. It has magnets which establish a field across the turning part called rotor. The magnets may be electromagnets or for small motors, permanent magnet. In an electromagnet motor, stator is wound with wire and current is forced through this winding called field winding. For a constant field current the magnetic flux 'ϕ' is constant, ϕ may be varied by varying the field current.

The rotor is wound with wire and through this winding called armature winding, a current is forced through the stati...

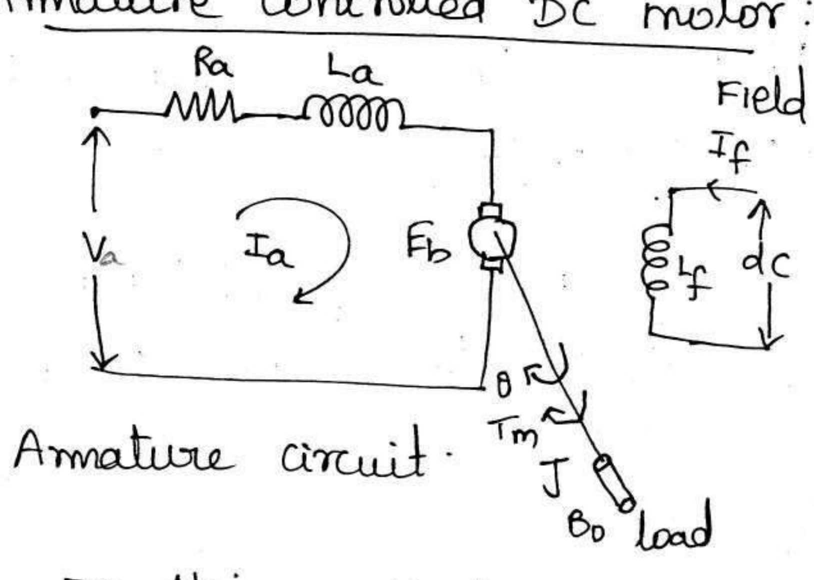
A DC servo motor is a specially designed DC motor with high starting torque and low inertia. A servo mechanism also called position control system, is a feedback control system and consists of a mechanism in which output of the system may be some mechanical position, velocity or acceleration.

DC motor are used in two different control modes.

a) Armature control mode with constant field current and

b) Field control mode with fixed armature current.

a) Armature controlled DC motor :- → * Armature current is varied.
* Field current is held constant



* I_a → armature current
E_b → back emf
J → moment of inertia
B_o → dashpot
L_f → Field
T_m → Torque
R_a → armature Resistance
V_a → armature voltage
ω_m → Angular velocity

In this method, Speed is varied by changing the armature Voltage keeping the field flux constant. The motor has rotating armature with load on its Shaft having a moment of inertia J and viscous friction coefficient B by motor is T_m which causes rotation. The angular velocity is ω and angular acceleration is α. Torque of motor is due to armature current I_a and field flux ϕ. again ϕ is proportional to I_f.

$$T_m = K_1 \phi I_a$$

$\phi \propto i_f$

$\phi = K_f i_f$

$$T_m = K_1 K_f I_f I_a$$

$$① \leftarrow \boxed{T_m = K_T I_a} \quad \text{where } K_T = K_1 K_f I_f$$

$\omega = \frac{d\theta}{dt}$

Back emf, $E_b = K\phi N = \frac{K\phi \omega}{2\pi} = \frac{K K_1 I_f}{2\pi} \frac{d\theta}{dt} = K_b \frac{d\theta}{dt} \rightarrow ②$ (angular velocity)

Inertia torque = Moment of inertia × Angular Acceler.

$$T_i = J \cdot \frac{d^2\theta}{dt^2} \rightarrow ③$$

Friction torque = Viscous Friction Coeff × Ang. Velocity

$$T_f = B_0 \omega = B_0 \frac{d\theta}{dt} \rightarrow ④$$

Torque developed by motor is opposed by inertia torque and frictional torque. Thus from Newtons II law of rotational motion,

$$T_i + T_f = T_m \quad \leftarrow \text{Torque-load equation}$$

$$J \cdot \frac{d^2\theta}{dt^2} + B_0 \frac{d\theta}{dt} = T_m = K_T I_a \rightarrow ⑤$$

The differential equation involving quantities of armature circuit can be written as

$$L_a \cdot \frac{dI_a}{dt} + R_a I_a + E_b = V_a$$

Taking Laplace Transform of equation 2, 5 & 6.

$$E_b(s) = K_b s \theta(s) \longrightarrow 7$$

$$(Js^2 + B_0 s)\theta(s) = K_T I_a(s) = T_m(s) \longrightarrow 8$$

$$(L_a s + R_a) I_a(s) = V(s) - E_b(s) \longrightarrow 9$$

From eqn 8 & 9

$$[Js^2 + B_0 s]\theta(s) = K_T \frac{V(s) - E_b(s)}{L_a s + R_a}$$

using eqn 2 in the above,

$$[Js^2 + B_0 s][L_a s + R_a]\theta(s) = K_T V(s) - K_b K_T s \theta(s)$$

$$\left[(Js^2 + B_0 s)(L_a s + R_a) + K_T K_b s\right]\theta(s) = K_T V(s)$$

$$\text{Transfer function} = \frac{\theta(s)}{V(s)} = \frac{K_T}{(Js^2 + B_0 s)(L_a s + R_a) + K_T K_b s}$$

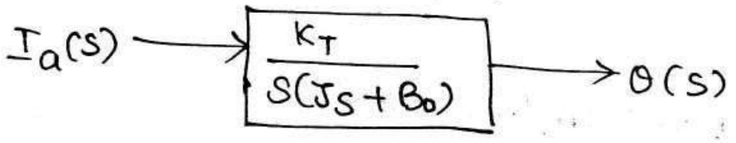
$$T.F = \frac{V_T}{s[(Js + B_0)(L_a s + R_a) + K_T K_b]}$$

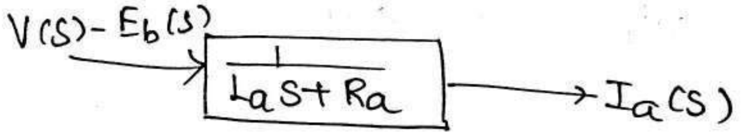
Block Diagram Representation,

From eq 9, we have $\dfrac{I_a(s)}{V(s) - E_b(s)} = \dfrac{1}{L_a s + R_a} \longrightarrow 10$

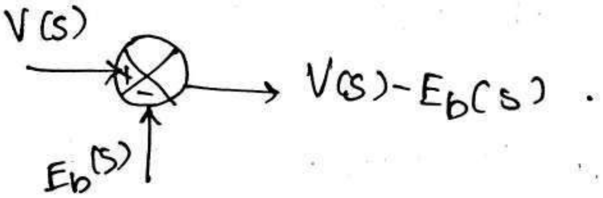
From eq 8, we have $\dfrac{\theta(s)}{I_a(s)} = \dfrac{K_T}{s(Js + B_0)} \longrightarrow 11$

eq (10) & (11) can be represented by,



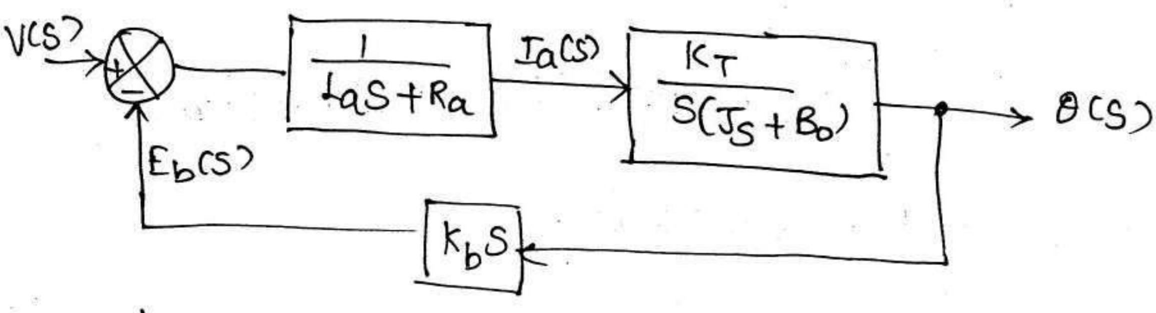


with V(S) as input signal and $E_b(S)$ as feedback signal. the component is represented as,

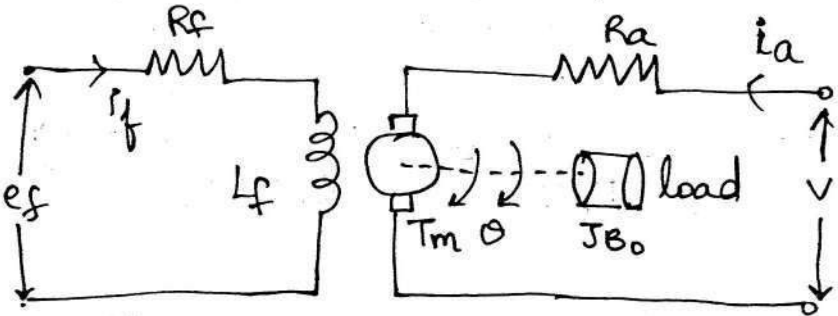


Again eq (7), $\boxed{E_b(S) = K_b S \theta(S)}$

combining all the above blocks, it will become.



ii) Field controlled DC Motor:-



$$T_m = K_1 \phi i_a = K_1 K_f i_f i_a = K i_f$$

$$\boxed{K = K_1 K_f i_a}$$

Also field circuit equation,

$$e_f = L_f \frac{di_f}{dt} + R_f i_f \longrightarrow (1)$$

Torque equation is,

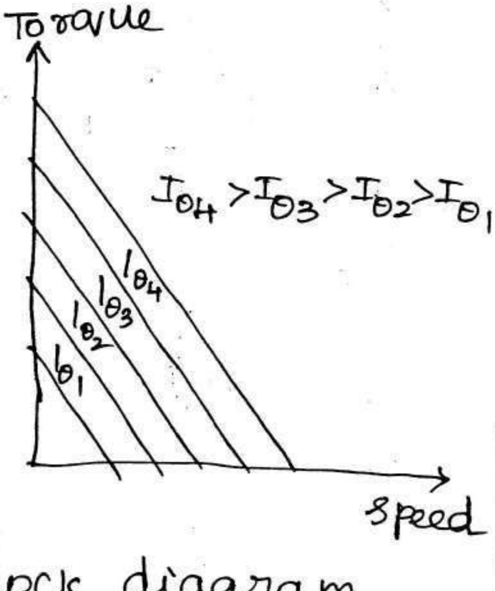
$$J\frac{d^2\theta}{dt^2} + B_0 \frac{d\theta}{dt} = T_m = K i_f \longrightarrow (2)$$

Taking Laplace transform on above eqns.

$$(L_f s + R_f) I_f(s) = E_f(s) \longrightarrow (3)$$

$$(Js^2 + B_0 s)\theta(s) = K I_f(s) \longrightarrow (4)$$

$$T.F = \frac{\theta(s)}{E_f(s)} = \frac{K}{s(Js+B_0)(L_f s+R_f)}$$



From eq (3), $\dfrac{I_f(s)}{E_f(s)} = \dfrac{1}{L_f s + R_f}$

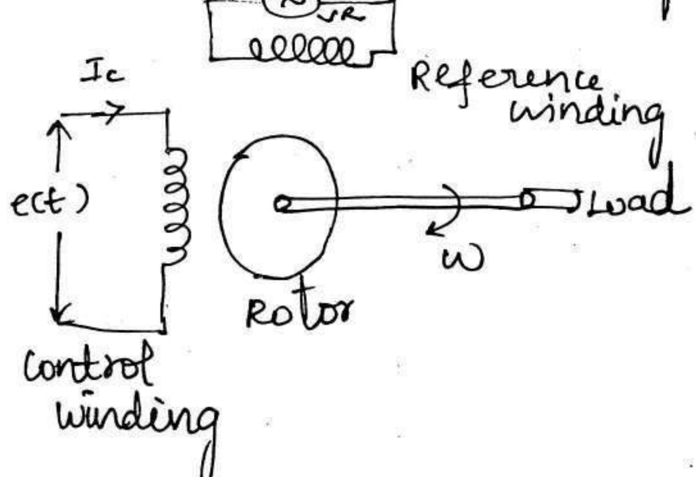
From eq (4), $\dfrac{\theta(s)}{I_f(s)} = \dfrac{K}{Js^2 + B_0 s}$.

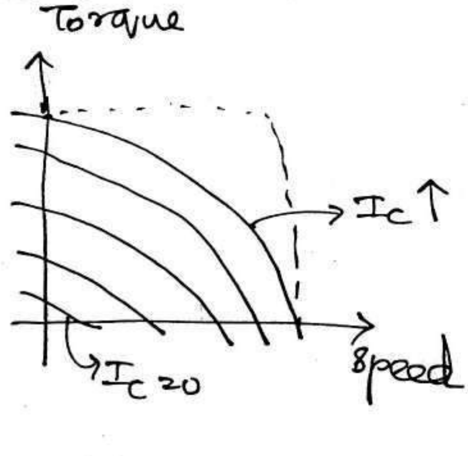
combining all the eqns in block diagram,

$$E_f(s) \longrightarrow \boxed{\frac{1}{L_f s + R_f}} \xrightarrow{I_f(s)} \boxed{\frac{K}{s(Js + B_0)}} \longrightarrow \theta(s)$$ //

AC SERVO MOTOR:-

AC servo motors are two phase induction motor suitable for simple and low power applications. It has Stator having two field windings placed at right angles to each other in order to produce a rotating field on which motor action depends. One phase is supplied from constant AC reference voltage V_R, the other phase act as a controlling field and is supplied from output of servo amplifier. The speed of rotation is proportional to control current I_c, the phase of which determines the direction of rotation.





It can be seen that curve for zero control current goes through origin and the slope is negative when control current becomes zero, the motor develops a deceleration torque, causing it to stop. The curve also shows a large torque at zero speed at increased I_c. For normal induction motor the ratio of X_2/R_2 is high. But for servo motors, X_2/R_2 ratio is kept low to obtain linear torque-speed characteristics. High value of R_2 will ensure high starting torque.

$$T_M = f(\theta, E)$$

$$T_M = T_{M0} + k(E - E_0) - f(\dot{\theta} - \dot{\theta}_0) \rightarrow (1)$$

where $k = \left.\frac{\delta T_m}{\partial E}\right|_{E = E_0, \dot{\theta} = \theta_0}$

$$f = \left.-\frac{\delta T_m}{\partial \dot{\theta}}\right|_{E = E_0,\ \dot{\theta} = \dot{\theta}_0}$$

from (1), $T_M - T_{M0} = k(E - E_0) - f(\dot{\theta} - \dot{\theta}_0)$

$$\Delta T_M = k\Delta_E - f\Delta\dot{\theta}$$

where $(\Delta T_M = T_M - T_{M0})$ and so on.

If J and f_0 be the inertia and viscous friction coefficient respectively of the load, the torque equation becomes,

$$\Delta T_M = J\Delta\ddot{\theta} + f_0\Delta\dot{\theta} = k\Delta E - f\Delta\dot{\theta}$$

Taking Laplace Transform,

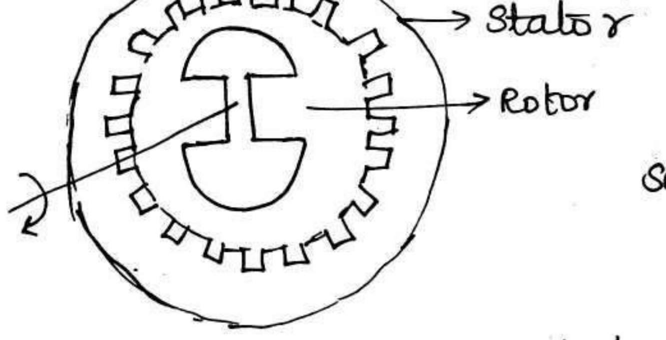
$$(Js^2 + f_0 s)\Delta\theta(s) = k \Delta E(s) - fs\Delta\theta(s).$$

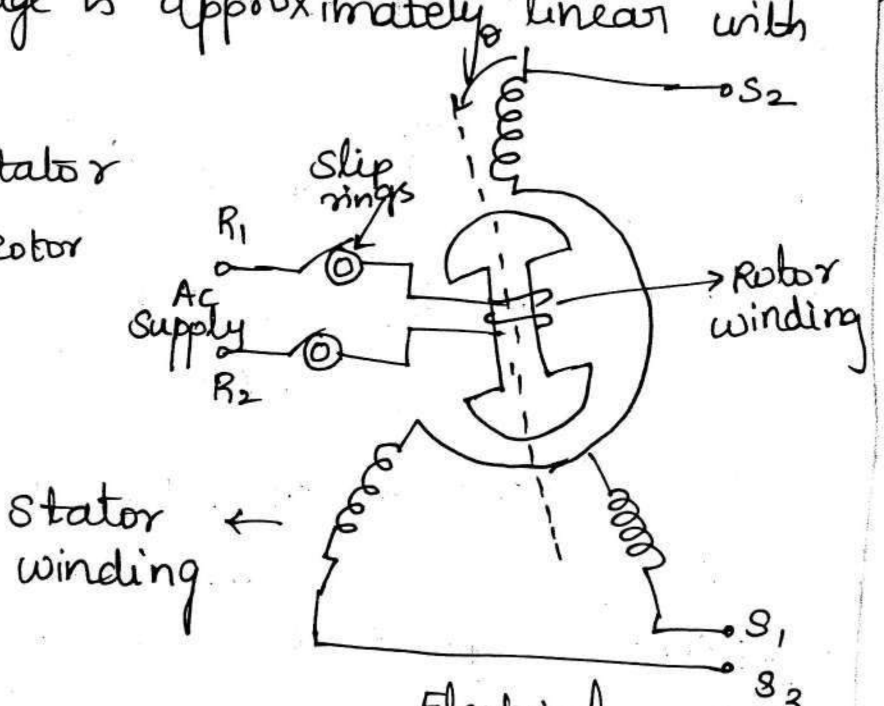
when an AC motor is used in position control system, operating point becomes $E_0 = 0$, $\dot{\theta}_0 = 0$, $\Delta\theta = \theta$, $\Delta E = E$.

$$G(s) = \frac{\theta(s)}{E(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

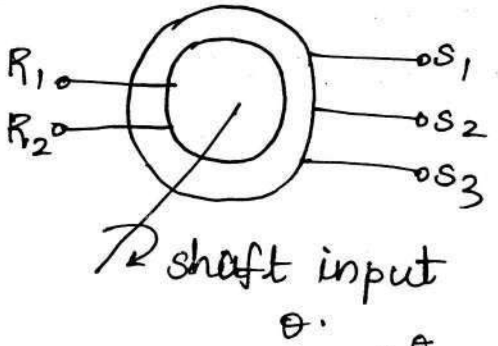
Synchros

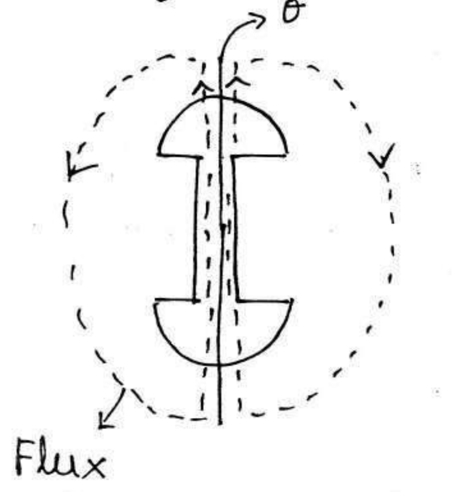
A synchro system formed by interconnection of the devices called synchro transmitter and the synchro control transformer. It is most widely used for error detector in feedback control systems. It measures and compares two angular displacements and its output voltage is approximately linear with angular difference.

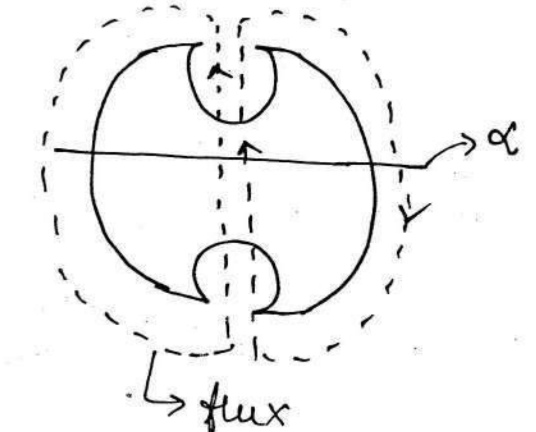




Electrical circuit







The stationary part of machine (stator) is slotted to accomodate three Y connected coils wound with their axes 120° apart. The stator windings are not directly connected to the ac power source. Their excitation is supplied by ac magnetic field produced by the rotor.

The rotor of a dumb-bell construction with a single winding. A single phase excitation voltage is applied to the rotor through two slip rings.

The resultant current produces a magnetic field and by the transformer action, induces voltages in the stator coils. The effective voltage induced in any stator coil depends up on angular position of coil axis with respect to the rotor axis

It offer higher sensitivity, longer life, ruggedness and continous rotation capability.

Since three stator windings of synchro transmitter are connected respectively in parallel with three stator windings of control transformer, stator winding induced voltage of synchro transmitter would cause current to flow and produce a resultant flux in the air-gap of synchro control transformer, which in turn will induce an emf in rotor. The magnitude of induced emf would be proportional to θ_r and θ ...

At 90°, the induced emf will be maximum.
Infact, the magnitude of induced emf at o/p, ie., across rotor terminals, will have relation,

$$e = K_s \sin(\theta_r - \theta_o)$$

$$e = K_s (\theta_r - \theta_o)$$ (∵ $\sin\theta \simeq \theta$ if θ is small)

$$\boxed{e = K_s \theta_e}$$

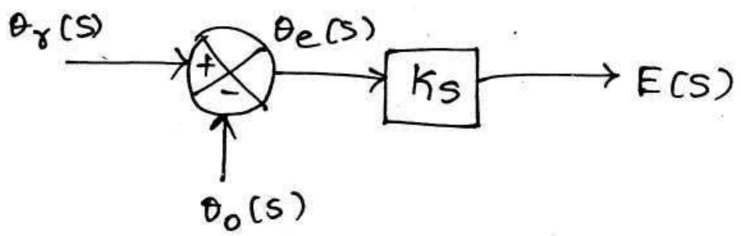
$K_s \rightarrow$ Proportionality Constant.

Taking Laplace Transform,

$$E(s) = K_s[\theta_r(s) - \theta_o(s)]$$

$$\boxed{E(s) = K_s\ \theta_e(s)}$$

Block Diagram :-



$$\boxed{K_s = \frac{E(s)}{\theta_e(s)} = \frac{E(s)}{\theta_r(s) - \theta_o(s)}}$$

where K_s is known as the sensitivity or the gain of error detector.

Synchros are used widely in control systems as detectors and encoders because their rigidness in construction and high reliability.

— x — x —

Multivariable control systems

control system

based on command & o/p.

SISO — MIMO

SISO:

one input affect primarily one output and has only weak effect on other outputs. It is possible to ignore weak interactions (coupling) and design controllers under the assumption that one input affect only one output.

MIMO:

It consists of appropriate number of separate SISO systems. coupling effects are considered as disturbance to separate control systems and may not cause significant degradation in their performance if the coupling is weak.

If it have strong interaction (coupling) if one input affect more than one output appreciably. There are two approaches for design of controllers for each system.

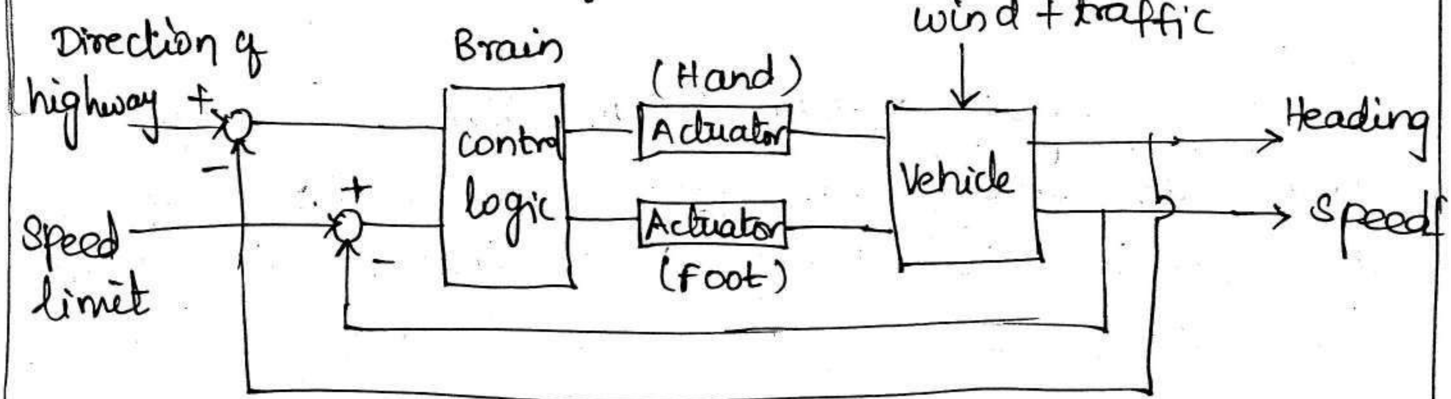
(i) Design a decoupling controller to cancel the interaction inherent in the system.

(ii) Design a single controller for multivariable

Examples of Multivariable control system are
(i) Automobile Driving System.
(ii) Antenna stabilization system.

It can be illustrated as follows.

(i) Automobile Driving System:



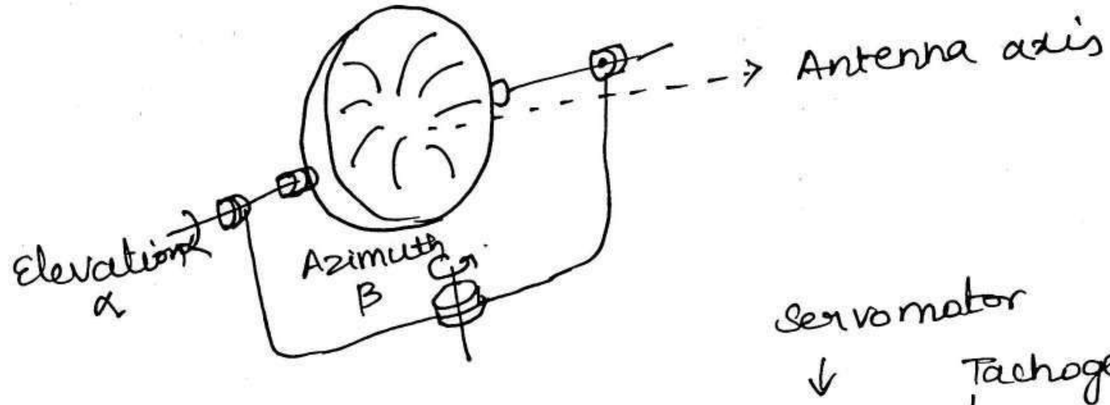
The system to be controlled has two inputs (steering & Acceleration/braking) and has two controlled outputs (heading & speed). Automobile driving system is a MIMO System. It can decouple this system in to two SISO system for the purpose of design.

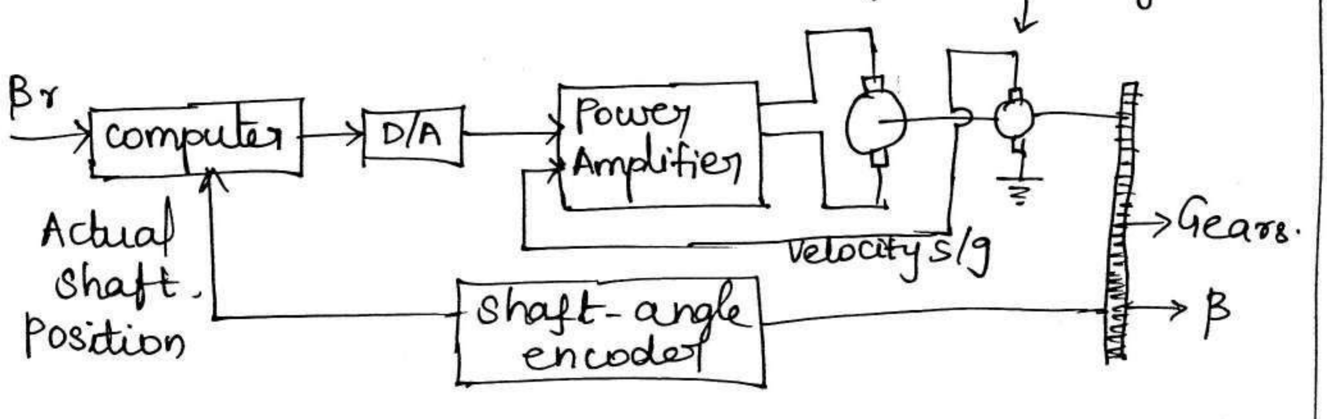
Steering control affects the heading and not speed and the accelerator control affect the speed and not heading. However brake of vehicle for speed control decreases the side forces at the tyre-road interface for directional control and with locked wheels the directional control is completely lost.

The command inputs cannot be constant set points. These inputs depends on traffic and road conditions and vary in an uncontrolled manner.

The actual signal points to the system are derived by driver from actual road and traffic conditions. The human operator ~~subsystem~~ will therefore be a component in the overall control system of the automobile. human operator then controls the manipulated Variables in a manner which reduces absolute error.

(ii) Servomechanism for steering of Antenna:-





The control system has for steering of an antenna can be treated as two independent s/m.

a) Azimuth angle servomechanism.

b) elevation angle servomechanism.

This is because interaction affect are

The occurence of azimuth angle error angle is ~~comp~~ causes an error signal to pass through amplifier, which increases the angular velocity of the servomotor in a direction towards an error reduction.

The main disturbance input is the deviation of load from the nominal estimated value as a result of uncertainty in our estimate, effect of wind power etc.,

Here windpower is disturbance in controlled system. Target surface fluctuations is a measurement noise.

These are the examples for Multi-Variable control systems.

— x — x —

UNIT: II TIME RESPONSE ANALYSIS:

Transient & steady state response - Measures of performance of first order & second order system - Effect of additional pole & additional zero - steady state error constant & system - Type number - PID controller - Analytical design of PI, PD, PID controller.

— x — x —

Introduction:

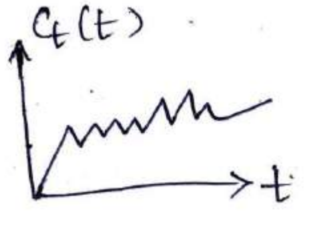
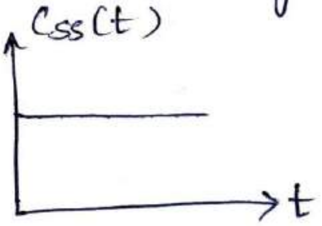
Time response analysis is also called time domain analysis. ie., output as a function of time.

Total time response $c(t)$ of a control system consists of transient response (dynamic response) $c_t(t)$ and steady state response $c_{ss}(t)$:

$$\boxed{c(t) = c_t(t) + c_{ss}(t)}$$

→ steady state response
→ Transient response
→ Total time response.

A feedback control system has the inherent capabilities that its parameters can be adjusted to alter both its transient and steady-state behaviour.

Transient Response	Steady state Response
It remains for a very short time. 	It remains as time 't' approaches infinity (longtime) 
It depends on system poles only. not on type of input.	It depends on both system poles and type of input.

Before proceeding with time response analysis of a control system, it is necessary to test stability of the system through indirect tests without actually obtaining the transient response. In case, system is unstable. hence of no practical use.. we need not proceed with its transient response Analysis.

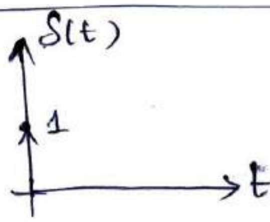
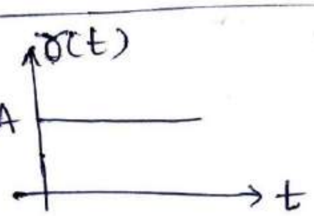
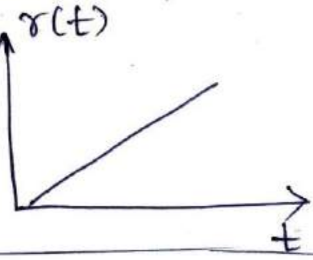
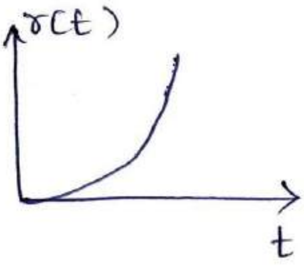
Typically test signals generated in laboratory are,

(i) unit Impulse (sudden shock)

(ii) unit step (sudden change)

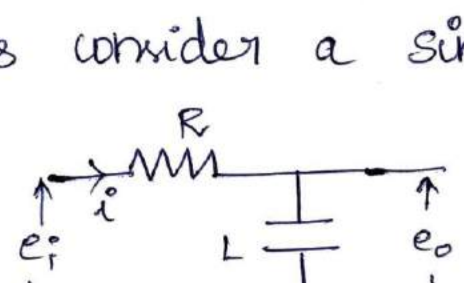
(iii) Ramp (constant velocity)

(iv) Parabolic (constant acceleration)

Sno	Signal	Diagram	Input $r(t)$	output $R(s)$
1	unit Impulse		$r(t)=\delta(t) = \begin{cases} 1, & t=0 \\ 0, & t\neq 0 \end{cases}$	$R(s) = 1$
2.	unit step		$r(t) = u(t) = \begin{cases} A, & t>0 \\ 0, & t<0 \end{cases}$	$R(s) = \frac{1}{s}$
3.	Ramp		$r(t)=\begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$	$R(s) = \frac{1}{s^2}$
4.	parabolic signal		$r(t)=\begin{cases} \frac{t^2}{2}, & t\geq 0 \\ 0, & o.w \end{cases}$	$R(s) = 1/s^3$

Measures of performance of First order system.

Let us consider a simple RC circuit,.



RC circuit a first order s/m.

$$e_i = Ri + \frac{1}{C}\int i\,dt \longrightarrow (1)$$

$$e_o = \frac{1}{C}\int i\,dt \longrightarrow (2)$$

Taking Laplace transform for (1) & (2)

$$E_i(s) = \left[R + \frac{1}{Cs}\right] I(s) \longrightarrow (3)$$

$$E_o(s) = \frac{1}{Cs} I(s) \longrightarrow (4)$$

Transfer function: $\frac{E_o(s)}{E_i(s)} = \frac{(1/Cs)\,\cancel{I(s)}}{(R + 1/Cs)\,\cancel{I(s)}}$

$$\frac{E_o(s)}{E_i(s)} = \frac{\cancel{Cs}}{(1+RCs)\,\cancel{Cs}} = \frac{1}{1+RCs} \longrightarrow (5)$$

where $\tau = RC$ = Time constant.

$$\therefore \boxed{T.F = \frac{1}{1+\tau s}}$$

$$\Rightarrow T.F = \frac{C(s)}{R(s)} = \frac{1}{1+\tau s}$$

$$\boxed{C(s) = R(s)\left(\frac{1}{1+\tau s}\right)} \longrightarrow \text{Main Equation.}$$

<u>Input : unit impulse s/g</u>

$$R(s) = 1$$

$$\therefore C(s) = \frac{1}{1+\tau s} \longrightarrow (6) \qquad C(s) = \frac{1}{\tau(s+1/\tau)}$$

Steady state error:-

$$e_{ss} = \lim_{t\to\infty}[e(t)] = \lim_{t\to\infty}[r(t) - c(t)]$$

[or]

$$e_{ss} = \lim_{s\to 0} s[E(s)] = \lim_{s\to 0} s[R(s) - C(s)] \rightarrow (8)$$

Substitute R(s) & C(s) value in (8).

$$\therefore e_{ss} = \lim_{s\to 0} s\left[1 - \frac{1}{\tau s+1}\right]$$

$$\boxed{e_{ss} = 0}$$

Input: unit step s/g

$R(s) = 1/s$ substitute in main eqn

$$\therefore C(s) = \left(\frac{1}{s}\right)\left(\frac{1}{1+\tau s}\right)$$

$$\frac{1}{s(1+\tau s)} = \frac{A}{s} + \frac{B}{1+\tau s}$$

$1 = A(1+\tau s) + Bs$ put $s=0$

$$\boxed{1 = A}$$

put $s = -1/\tau$

$$1 = B\left(\frac{-1}{\tau}\right)$$

$$\boxed{B = -\tau}$$

$$\therefore \boxed{C(s) = \frac{1}{\ } - \frac{\tau}{\ }}$$

Apply inverse L.T on above eqn.

$$C(s) = \frac{1}{s} - \frac{\tau}{1+\tau s}$$

$$C(s) = \frac{1}{s} - \frac{\tau}{\tau(s+1/\tau)}$$

$$\boxed{c(t) = 1 - e^{-t/\tau}}$$

Steady state error:-

$$e_{ss} = \lim_{s \to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s} - \frac{1}{s(1+\tau s)}\right]$$

$$= \lim_{s \to 0} \left(1 - \frac{1}{1+\tau s}\right)$$

$$\boxed{e_{ss} = 0}$$

Input: unit Ramp signal.

$R(s) = 1/s^2$ ⇒ substitute in main eqn.

$$\therefore C(s) = \frac{1}{s^2}\left(\frac{1}{1+\tau s}\right)$$

$$C(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{1+\tau s}.$$

④

$$1 = \cancel{AS^2 + AS(\tau S+1) + CS^2}$$

$$1 = AS(\tau S+1) + B(1+\tau S) + CS^2$$ $\boxed{\text{Put } S=0}$

$$\boxed{1 = B}$$

$\boxed{\text{Put } S = -1/\tau}$

$$1 = C\left(-1/\tau\right)^2$$

$$\boxed{C = \tau^2}$$

$\boxed{\text{Put } S=1}$

$$1 = A(1+\tau) + B(1+\tau) + C$$

$$1 = (A+B)(1+\tau) + C$$

$$1 = (A+1)(1+\tau) + \tau^2$$

$$\cancel{1} = A\cdot(1+\tau) + \cancel{1} + \tau + \tau^2$$

$$-\tau(\cancel{1+\tau}) = A(\cancel{1+\tau})$$

$$\boxed{A = -\tau}$$

$$\therefore \quad C(S) = \frac{-\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{1+\tau S}$$

$$C(S) = \frac{-\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{\tau(S+1/\tau)}$$

Take I.L.T,

$$C(t) = -\tau + 1 + \tau e^{-t/\tau}$$

$$\boxed{C(t) = 1-\tau(1 - e^{-t/\tau})}$$

Steady state error :-

$$e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s\to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(\tau s+1)}\right]$$

$$= \lim_{s\to 0} \frac{1}{s}\left(1 - \frac{1}{\tau s+1}\right)$$

$$= \lim_{s\to 0} \frac{1}{\cancel{s}}\left(\frac{\tau \cancel{s}}{1+\tau s}\right)$$

$$\boxed{e_{ss} = \tau}$$

Response of first order system.

Sno	Signal	input $R(s)$	output $C(t)$	Steady state error e_{ss}
1.	Impulse signal	1	$\frac{1}{\tau} e^{-t/\tau}$ Diff	0
2.	Step signal	$1/s$	$1 - e^{-t/\tau}$ Integ.	0
3.	Ramp signal	$1/s^2$	$t - \tau(1-e^{-t/\tau})$	τ

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$\cancel{1 = AS^2 + AS(\tau S+1) + CS^2}$

$1 = AS(\tau S+1) + B(1+\tau S) + CS^2$ 	 $\boxed{\text{Put } S=0}$

$\boxed{1 = B}$

$\boxed{\text{Put } S = -1/\tau}$

$1 = C\left(-1/\tau\right)^2$

$\boxed{C = \tau^2}$

$\boxed{\text{Put } S=1}$

$1 = A(1+\tau) + B(1+\tau) + C$

$1 = (A+B)(1+\tau) + C$

$1 = (A+1)(1+\tau) + \tau^2$

$\cancel{1} = A\cdot(1+\tau) + \cancel{1} + \tau + \tau^2$

$-\tau(1+\cancel{\tau}) = A(1\cancel{+\tau})$

$\boxed{A = -\tau}$

$\therefore \quad C(S) = -\frac{\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{1+\tau S}$

$C(S) = -\frac{\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{\tau(S+1/\tau)}$

Take I.L.T,

$C(t) = -\tau + 1 + \tau e^{-t/\tau}$

$\boxed{C(t) = 1 - \tau(1 - e^{-t/\tau})}$

Differentiation of step signal $\Rightarrow$ Impulse s/g.

$$\frac{d}{dt}(1-e^{-t/\tau}) \Rightarrow \frac{1}{\tau}e^{-t/\tau}.$$

Integration of step signal $\Rightarrow$ Ramp signal.

$$c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} + C.$$

Assume initial conditions are zero,

$$c(0) = 0 + \tau + C.$$

$$\boxed{C = -\tau}$$

$$\therefore c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} - \tau \Rightarrow \text{Ramp s/g}.$$

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Measures of performance of Second order system.

The standard form of closed loop transfer function of second order system is given by,

$$T.F = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $\omega_n \rightarrow$ Natural undamped freq (rad/sec)

$\xi \rightarrow$ damping ratio $= \frac{\text{Actual damping}}{\text{critical damping}}$ (no units)

Depend on Damping ratio 'ξ', system can be classified in to four cases:

(i) undamped system : $\xi = 0$

(ii) under damped system : $0 < \xi < 1$

(iii) critically damped system : $\xi = 1$

(iv) over damped system : $\xi > 1$

The characteristic equation of second order system,

$$\boxed{s^2 + 2\xi\omega_n s + \omega_n^2 = 0}$$

It is a Quadratic equation, & root of this eqn,

$$\text{Quadratic formula} \Big\} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$a = 1,\ b = 2\xi\omega_n,\ c = \omega_n^2$

$$\therefore \text{roots } s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}} \Rightarrow \text{roots of second order eqn.}$$

case (i) undamped ($\xi = 0$)

$$\boxed{s_1, s_2 = \pm j\omega_n} \text{ (pure imaginary)}$$

Case (ii) under damped $0 < \xi_p < 1$

$$s_1, s_2 = -\xi_p\omega_n \pm \omega_n\sqrt{\xi_p^2 - 1}$$

$$= -\xi_p\omega_n \pm \omega_n\sqrt{(-1)(1-\xi_p^2)}$$

$$\boxed{s_1, s_2 = -\xi_p\omega_n \pm j\,\omega_n\sqrt{1-\xi_p^2}}$$ (complex conjugate).

Case (iii) critically damped $\xi_p = 1$

$$\boxed{s_1, s_2 = -\omega_n}$$ (real & equal).

Case iv Over damped $\xi_p > 1$

$$\boxed{s_1, s_2 = -\xi_p\omega_n \pm \omega_n\sqrt{\xi_p^2 - 1}}$$ (real & unequal)

In general, roots of second order s/m $\Big\}$ $\boxed{s_1, s_2 = -\xi_p\omega_n \pm \omega_n\sqrt{\xi_p^2 - 1}}$

where $\sigma = \xi_p\omega_n \Rightarrow$ attenuation &

$\omega_n\sqrt{1-\xi_p^2} = \omega_d \Rightarrow$ frequency of damped oscillation (rad/sec).

$\therefore$ roots of Second order System $\Big\}$ $\boxed{s_1, s_2 = -\sigma \pm \omega_d}$ //.

Time Domain response of second order system for unit Impulse signal.

Second order Transfer fn $\boxed{T = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$

Input : impulse s/g. $\therefore$ $C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$\hookrightarrow$ Main eqn.

Case (i): undamped s/m ($\xi = 0$) sub in main eqn,

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} = \omega_n \cdot \frac{\omega_n}{s^2 + \omega_n^2}$$

Taking Inverse L.T,

$$\boxed{c(t) = \omega_n \sin \omega_n t}$$

Case (ii): critically damped s/m ($\xi = 1$) sub in main eqn.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

Taking I.LT

$$\boxed{c(t) = \omega_n^2 \, t \, e^{-\omega_n t}}$$

<u>case (iii)</u>: Overdamped system ($\xi_p > 1$) sub in main eq$_n$.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi_p \omega_n s + \omega_n^2}$$

<u>Roots</u> $s_1, s_2 = -\xi_p \omega_n \pm \omega_n \sqrt{\xi_p^2 - 1}$.

$$s_1 = -\xi_p \omega_n + \omega_n \sqrt{\xi_p^2 - 1}$$

$$s_2 = -\xi_p \omega_n - \omega_n \sqrt{\xi_p^2 - 1}$$

$$\therefore C(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2} = \frac{\omega_n^2}{s^2 + 2\xi_p \omega_n s + \omega_n^2}$$

$\omega_n^2 = A(s - s_2) + B(s - s_1)$ sub $\boxed{s = s_1}$

$\omega_n^2 = A(s_1 - s_2)$

$$\boxed{A = \frac{\omega_n^2}{s_1 - s_2}}$$ substitute s_1 & s_2 values.

$$A = \frac{\omega_n^2}{\cancel{-\xi_p \omega_n} + \omega_n \sqrt{\xi_p^2 - 1} + \cancel{\xi_p \omega_n} + \omega_n \sqrt{\xi_p^2 - 1}}$$

$$= \frac{\omega_n^{\cancel{2}}}{2 \cancel{\omega_n} \sqrt{\xi_p^2 - 1}}$$

$$\boxed{A = \frac{\omega_n}{2\sqrt{\xi_p^2 - 1}}}$$

substitute $s = s_2$,

$$B = \frac{\omega_n^2}{s_2 - s_1}$$

similarly as like A,

$$\boxed{B = \frac{-\omega_n}{2\sqrt{\xi^2-1}}}$$

$$\therefore\ C(s) = \frac{\omega_n/2\sqrt{\xi^2-1}}{s-s_1} - \frac{\omega_n/2\xi^2-1}{s-s_2}.$$

$$C(s) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

Take inverse Laplace Transform,

$$\boxed{C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(e^{s_1 t} - e^{s_2 t}\right)}$$

Substitute s_1 & s_2,

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t} - e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t}\right]$$

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{-\xi\omega_n t}\left(e^{(\omega_n\sqrt{\xi^2-1})t} - e^{(-\omega_n\sqrt{\xi^2-1})t}\right]\right.$$

$$\boxed{C(t) = \frac{\omega_n}{\sqrt{\xi^2-1}}\, e^{-\xi\omega_n t}\left[\sinh\omega_n\sqrt{\xi^2-1}\right]t}$$

case(iv): underdamped system ($0 < \xi_p < 1$)

Roots: $S_1, S_2 \Rightarrow -\xi_p \omega_n \pm j\omega_n\sqrt{1-\xi_p^2}$.

$S_1 = -\xi_p\omega_n + j\omega_n\sqrt{1-\xi_p^2}$

$S_2 = -\xi_p\omega_n - j\omega_n\sqrt{1-\xi_p^2}$.

$$C(S) = \frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2} = \frac{\omega_n^2}{(S-S_1)(S-S_2)}$$

$$\frac{\omega_n^2}{(S-S_1)(S-S_2)} = \frac{A}{S-S_1} + \frac{B}{S-S_2}.$$

$\omega_n^2 = A(S-S_2) + B(S-S_1)$ if $(S = S_1)$

$\omega_n^2 = A(S_1-S_2)$

$$A = \frac{\omega_n^2}{S_1-S_2} = \frac{\omega_n^{\cancel{2}}}{2j\cancel{\omega_n}\sqrt{1-\xi_p^2}} = \boxed{\frac{\omega_n}{2j\sqrt{1-\xi_p^2}} = A}$$

Illly $\boxed{B = \frac{-\omega_n}{2j\sqrt{1-\xi_p^2}}}$.

$$\therefore C(S) = \frac{\omega_n/2j\sqrt{1-\xi_p^2}}{S-S_1} - \frac{\omega_n/2j\sqrt{1-\xi_p^2}}{S-S_2}.$$

$$C(s) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} \left(\frac{1}{s-s_1} - \frac{1}{s-s_2} \right)$$

Take I.L.T

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} \left(e^{s_1 t} - e^{+s_2 t} \right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} \left(e^{\left(-\xi\omega_n + j\omega_n\sqrt{1-\xi^2}\right)t} - e^{\left(-\xi\omega_n - j\omega_n\sqrt{1-\xi^2}\right)t} \right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cdot \left[e^{\left(j\omega_n\sqrt{1-\xi^2}\right)t} - e^{\left(-j\omega_n\sqrt{1-\xi^2}\right)t} \right]$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left[\omega_n\sqrt{1-\xi^2}\right]t \qquad \left(\because \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta \right.$$

$$\boxed{c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\right)t} \;//$$

<u>Steady state error</u>:- $e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right]$$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{\omega_n^2}\right] = 0 \;//$$

Time Response of second order system for unit step signal.

T.F of a second order system } $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2}$

unit step signal $\boxed{R(S) = 1/S}$

$$\therefore \; C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2}\right)$$

case (i) undamped, $\xi_p = 0$.

$$C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{S^2+\omega_n^2}\right) = \frac{A}{S} + \frac{BS+C}{S^2+\omega_n^2}$$

$$\omega_n^2 = A(S^2+\omega_n^2) + (BS+C)(S)$$

$S=0$

$$\omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A=1}$$

Expand the eqn,

$$\omega_n^2 = AS^2 + A\omega_n^2 + BS^2 + CS$$

Equating s^2 terms $\Rightarrow$ $0 = A+B$ $\therefore \boxed{B=-1}$

Equating s term $\Rightarrow \boxed{0 = C}$

$$\therefore \; C(S) = \frac{1}{S} + \left(\frac{-S}{S^2+\omega_n^2}\right) = \frac{1}{S} - \frac{S}{S^2+\omega_n^2}$$

Taking Inverse L.T,

$$\boxed{c(t) = 1 - \cos\omega_n t}$$

case ii critically damped system ($\xi = 1$)

$$C(S) = R(S)\left(\frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}\right)$$

$$C(S) = \left(\frac{1}{S}\right)\left(\frac{\omega_n^2}{S^2+2\omega_n S+\omega_n^2}\right) = \left(\frac{1}{S}\right)\left(\frac{\omega_n^2}{(S+\omega_n)^2}\right)$$

Apply partial fraction.

$$C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{(S+\omega_n)^2}\right) = \frac{A}{S} + \frac{B}{S+\omega_n} + \frac{C}{(S+\omega_n)^2}$$

$$\omega_n^2 = A(S+\omega_n)^2 + BS(S+\omega_n) + CS.$$

$$\omega_n^2 = A\omega_n^2 \quad \therefore \boxed{A=1} \qquad \text{put } S=0$$

Expand the equation.

$$\omega_n^2 = AS^2 + A\omega_n^2 + 2AS\omega_n + BS^2 + BS\omega_n + CS.$$

Equating s^2 term

$$0 = A + B \quad \therefore \boxed{B=-1}$$

Equating s term.

$$0 = 2A\omega_n + B\omega_n + C.$$

$$0 = 2\omega_n - \omega_n + C$$

$$\boxed{C = -\omega_n}$$

$$\boxed{A = 1; \ B = -1; \ C = -\omega_n}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Take inverse Laplace transform.

$$\boxed{c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}}$$

Case (iii) Overdamped system ($\xi > 1$)

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

roots $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$.

$$\therefore C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)}$$

Taking partial fraction.

$$C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{A}{s} + \frac{B}{s-s_1} + \frac{C}{s-s_2} \rightarrow (1)$$

$$\omega_n^2 = A(s-s_1)(s-s_2) + Bs(s-s_2) + Cs(s-s_1).$$

$$\omega_n^2 = A(-s_1)(-s_2) \quad \therefore \boxed{A = \frac{\omega_n^2}{s_1 s_2}} \qquad \text{put } s=0$$

Substitute s_1, s_2 values in (A) eqn

$$A = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(-\xi\omega_n - \omega_n\sqrt{\xi^2-1})}$$

$$A = \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)} \qquad (\because (a+b)(a-b) = a^2-b^2)$$

$$A = \frac{\cancel{\omega_n^2}}{\cancel{\xi^2\omega_n^2} - \cancel{\xi^2\omega_n^2} + \cancel{\omega_n^2}} = 1. \quad \therefore \boxed{A=1}$$

Put $s=s_1$, in partial fraction equation.

$$\omega_n^2 = B s_1(s_1-s_2)$$

$$\boxed{B = \frac{\omega_n^2}{s_1(s_1-s_2)}}$$

substitute s_1, s_2 values in (B) eqn.

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(\cancel{-\xi\omega_n} + \omega_n\sqrt{\xi^2-1} + \cancel{\xi\omega_n} + \omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(2\omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{1}{(-\xi + \sqrt{\xi^2-1})(2\sqrt{\xi^2-1})} \qquad \therefore \boxed{B = \frac{1}{2\sqrt{\xi^2-1}\,(-\xi+\sqrt{\xi^2-1})}}$$

substitute $s=s_2$ in partial fraction eqn.

$$\omega_n^2 = C s_2(s_2-s_1)$$

$$C = \frac{\omega_n^2}{}$$

$$C = \frac{\omega_n^2}{-\xi_\rho\omega_n - \omega_n\sqrt{\xi_\rho^2-1}\left[-2\omega_n\sqrt{\xi_\rho^2-1}\right]}$$

$$\boxed{C = \frac{1}{2\sqrt{\xi_\rho^2-1}\left(\xi_\rho+\sqrt{\xi_\rho^2-1}\right)}}$$

substitute A, B, C Values in eq(i)

$$C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi_\rho^2-1}\left(-\xi_\rho+\sqrt{\xi_\rho^2-1}\right)}\left(\frac{1}{s-s_1}\right) + \frac{1}{2\sqrt{\xi_\rho^2-1}\left(\xi_\rho+\sqrt{\xi_\rho^2-1}\right)}\left(\frac{1}{s-s_2}\right)$$

Taking inverse laplace Transform.

$$\boxed{c(t) = 1 + \frac{1}{2\sqrt{\xi_\rho^2-1}\left(-\xi_\rho+\sqrt{\xi_\rho^2-1}\right)}e^{s_1 t} + \frac{1}{2\sqrt{\xi_\rho^2-1}\left(\xi_\rho+\sqrt{\xi_\rho^2-1}\right)}e^{s_2 t}}$$ //

<u>case (iv)</u>: underdamped system ($\xi_\rho < 1$)

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi_\rho\omega_n s+\omega_n^2)}$$

Apply partial fraction,

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi_\rho\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi_\rho\omega_n s+\omega_n^2}$$

$$\omega_n^2 = A(s^2+2\xi_\rho\omega_n s+\omega_n^2) + (Bs+C)(s)$$

$s=0$

$$\omega_n^2 = A\,\omega_n^2 \Rightarrow \boxed{A=1}$$

Expand the partial fraction eqn.

$$\omega_n^2 = AS^2 + 2A\xi\omega_n S + A\omega_n^2 + BS^2 + CS.$$

Equating s^2 terms.

$$0 = A + B \quad \therefore \boxed{B = -1}$$

Equating s terms.

$$0 = 2A\xi\omega_n + C$$

$$0 = 2\xi\omega_n + C \quad \therefore \boxed{C = -2\xi\omega_n}$$

substitute A, B, C. Values.

$$C(s) = \frac{1}{s} + \frac{(-s + (-2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Add & subtract $\xi^2\omega_n^2$ in denominator of second term.

$$C(s) = \frac{1}{S} - \left(\frac{S + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2 + \xi^2\omega_n^2 - \xi^2\omega_n^2}\right)$$

$$= \frac{1}{S} - \left(\frac{S + 2\xi\omega_n}{(S + \xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right)$$

but $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore \boxed{\omega_d^2 = \omega_n^2(1-\xi^2)}$$

$$C(s) = \frac{1}{s} - \left(\frac{s + \xi\omega_n + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} \right)$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n(\omega_d)}{(\omega_d)\left((s+\xi\omega_n)^2 + \omega_d^2\right)}$$

Taking Inverse Laplace Transform.

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t} \sin\omega_d t \, .$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right].$$

substitute $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_n\sqrt{1-\xi^2})t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_n\sqrt{1-\xi^2}t\right]$$

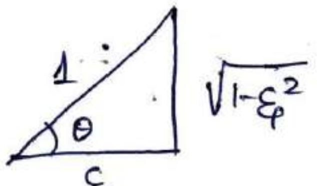
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[(\sqrt{1-\xi^2})\cos\omega_n\sqrt{1-\xi^2}t + \xi\sin(\omega_n\sqrt{1-\xi^2})t\right]$$

Rearrange the terms,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\xi\sin\omega_d t + \sqrt{1-\xi^2}\cos\omega_d t\right).$$

On constructing right angle triangle with ξ_p and $\sqrt{1-\xi_p^2}$ we get,



$$\therefore \sin\theta = \sqrt{1-\xi_p^2}$$
$$\cos\theta = \xi_p$$
$$\tan\theta = \frac{\sqrt{1-\xi_p^2}}{\xi_p} \Rightarrow \theta = \tan^{-1}\frac{\sqrt{1-\xi_p^2}}{\xi_p}$$

let us express $c(t)$ in standard form,

$$c(t) = 1 - \frac{e^{-\xi_p\omega_n t}}{\sqrt{1-\xi_p^2}}\left[(\sin\omega_d t \times \cos\theta) + (\cos\omega_d t \times \sin\theta)\right]$$

$$= 1 - \frac{e^{-\xi_p\omega_n t}}{\sqrt{1-\xi_p^2}}\sin(\omega_d t + \theta)$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\xi_p^2}}{\xi_p}$ $\quad\therefore \boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

$$\therefore \boxed{c(t) = 1 - \frac{e^{-\xi_p\omega_n t}}{\sqrt{1-\xi_p^2}}\sin(\omega_d t + \theta)}$$

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<u>Steady state error of second order s/m with unit step</u>

$$e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$$
$$= \lim_{s\to 0} s\left[\frac{1}{s} - \frac{\omega_n^2}{s(s^2+2\xi_p\omega_n s+\omega_n^2)}\right]$$
$$= \lim \left[1 - \frac{\omega_n^2}{\ldots}\right] = 0$$

SUMMARY

SECOND ORDER SYSTEM WITH IMPULSE INPUT

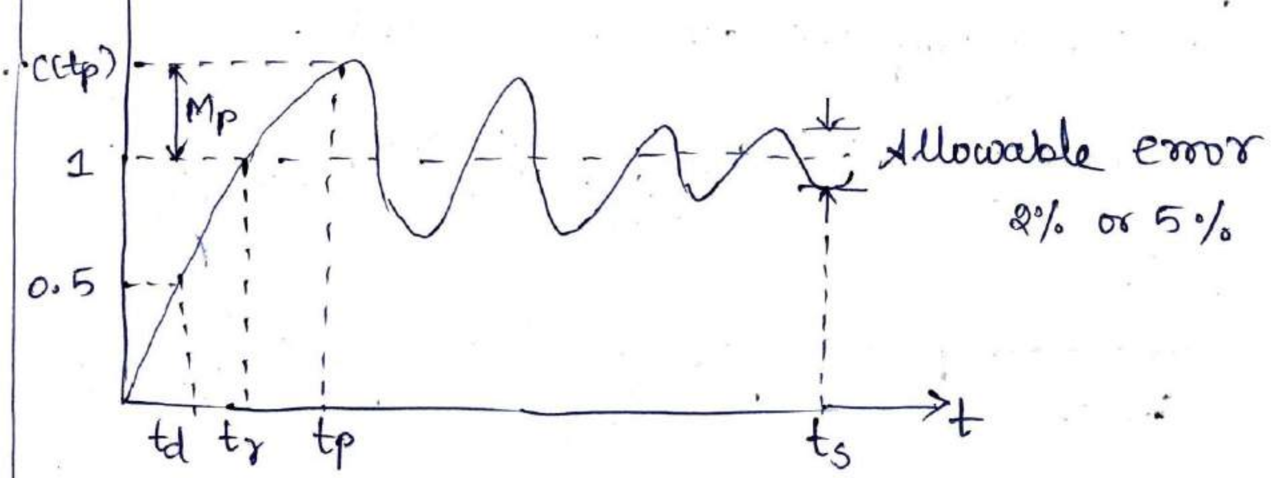
Sno	Damping Type	c(t)
1.	undamped ($\xi=0$)	$c(t) = \omega_n \sin\omega_n t$
2.	critically damp ($\xi=1$)	$c(t) = \omega_n^2 t e^{-\omega_n t}$
3.	overdamped ($\xi>1$)	$c(t) = \frac{\omega_n}{\sqrt{\xi^2-1}} e^{-\xi\omega_n t} \left[\sinh \omega_n\sqrt{\xi^2-1}\right]t$
4.	underdamped ($0<\xi<1$)	$c(t) = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n\sqrt{1-\xi^2})t$

SECOND ORDER SYSTEM WITH UNIT STEP INPUT

Sno	Damping Type	c(t)
1.	undamped ($\xi=0$)	$c(t) = 1 - \cos\omega_n t$
2.	critically damp ($\xi=1$)	$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$
3.	overdamped ($\xi>1$)	$c(t) = 1 + \frac{1}{2\sqrt{\xi^2-1}(-\xi+\sqrt{\xi^2-1})} e^{S_1 t} + \frac{1}{2\sqrt{\xi^2-1}(\xi+\sqrt{\xi^2-1})} e^{S_2 t}$
4.	underdamped ($0<\xi<1$)	$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$

Transient Response Specifications.

The transient response of a system to a unit step input depends on initial conditions. ∴ to compare the time response of Various System, it is necessary to start with standard initial conditions.



where, $t_d \rightarrow$ delay time

$t_r \rightarrow$ Rise time.

$t_p \rightarrow$ Peak time

$M_p \rightarrow$ Maximum peak overshoot

$t_s \rightarrow$ settling time.

The time domain specifications are defined as follows,

(i) Delay time : (t_d) :

It is the time required by the response to reach half of its final value at the

(ii) Rise time :(t_r):

It is the time required for response to rise from 10% to 90% for overdamped or critically damped s/m and 0% to 100% for underdamped of its final value. ~~the~~ 5% to 95% of its ~~its~~ final value for may be used for critically damped system.

In general, unit step response of second order for underdamped is given by,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Rise time $\Rightarrow \boxed{t = t_r}$ $\boxed{c(t_r) = 1}$

$$c(t_r) = \cancel{1} - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = \cancel{1}$$

The term $\sin(\omega_d t_r + \theta) = 0 \Rightarrow \sin\phi = 0$.

when $\phi = 0, \pi, 2\pi \ldots$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}} = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$ unit is degree.

but 'π' ...

(iii) Peak time: (t_p) :-

Peak time is obtained by differentiating c(t) with respect to t and equating to zero. At maxima, the slope is zero.

$$\therefore t_p = \left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0$$

(After differentiate c(t))

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

"The time required for the response to reach first peak of overshoot".

where $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$

(iv) Maximum peak overshoot: (M_p)

It is maximum peak value of the response measured from unity.

$$\therefore M_p = C(t_p) - 1 = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\% M_p = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100$$

$C(t_p) \rightarrow$ peak value of c(t)

$C(\infty) \rightarrow$ final value of c(t).

$$\therefore \boxed{M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}}$$

(v) Settling time: (t_s):

It is the time required for response curve to reach and stay within a specified tolerance band (either 2% or 5%) of final value.

Settling time t_s ~~for 2% tolerance band~~

$$t_s = \frac{4}{\xi_p \omega_n} \text{ for 2\% tolerance band \& } \boxed{\xi_p = 0.76}$$

$$t_s = \frac{3}{\xi_p \omega_n} \text{ for 5\% tolerance band \& } \boxed{\xi_p = 0.68}$$

—— × —— × ——

Problems on Response of the system:

1. Obtain the response of unity feedback system whose open loop transfer function is $G(S) = \frac{4}{S(S+5)}$ and when input is unit step.

= Solution :-

(Always prefer negative feedback)

For unity feedback,

$$T.F = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)} = \frac{4/S(S+5)}{1 + 4/S(S+5)}$$

$$\frac{C(S)}{R(S)} = \frac{4}{S^2+5S+4} = \cancel{\frac{A}{S+4}}$$

$C(S) = R(S) \left(\frac{4}{\ldots} \right)$

$$C(s) = \frac{4}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1}.$$

$$4 = A(s+4)(s+1) + B(s)(s+1) + C(s+4)s.$$

Put $s=0 \Rightarrow \boxed{A=1}$

$s=-4 \Rightarrow \boxed{B=1/3}$

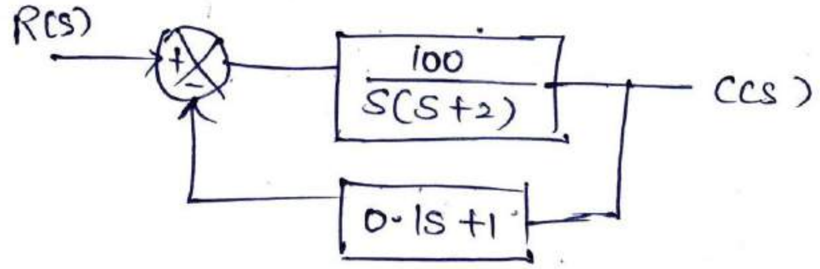
$s=-1 \Rightarrow \boxed{C=-4/3}$

$$\therefore C(s) = \frac{1}{s} + \frac{1}{3(s+4)} - \frac{4}{3(s+1)}$$

Take inverse Laplace transform,

$$\boxed{c(t) = 1 + \frac{1}{3}e^{-4t} - \frac{4}{3}e^{-t}}.$$

2. A position control system with velocity feedback is shown in figure. Calculate rise time, peak time, peak overshoot, settling time and also sketch the response.



= solution:-

$$T.F = \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

$$\frac{C(S)}{R(S)} = \frac{100/S(S+2)}{1+\left(100/S(S+2)\right)(0.1S+1)}$$

$$\boxed{\frac{C(S)}{R(S)} = \frac{100}{S^2+12S+100}} \longrightarrow ①$$

General formula for 2nd order system,

$$\boxed{\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}} \longrightarrow ②$$

compare ① & ②, $\therefore \omega_n^2 = 100 \Rightarrow \boxed{\omega_n = 10 \text{rad/sec}}$

$2\xi\omega_n = 12 \Rightarrow \boxed{\xi = 0.6}$ no unit

(i) rise time :-

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$\theta = \tan^{-1}\left(\sqrt{1-\xi^2}/\xi\right)$$

$$\theta = \tan^{-1}\left(\sqrt{1-0.6^2}/0.6\right) = \tan^{-1}(1.333) = 53.13^\circ$$

$\theta = 53.13 \times \frac{\pi}{180} = 0.92$ radian.

$\omega_d = \omega_n\sqrt{1-\xi^2} = 8$ rad/sec $\quad \therefore t_r = \pi - 0.92$ ... sec

(ii) peak time: $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{8} = 0.39 \text{ sec}$.

(iii) Max. peak overshoot: $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$= e^{-0.6\pi/\sqrt{1-0.6^2}}$$

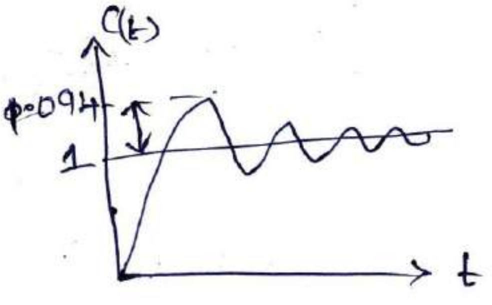
$$\boxed{\begin{aligned} M_p &= 0.094 \\ \% M_p &= 9.4\% \end{aligned}}$$

(iv) settling time (t_s) :-

$$2\%\ t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6 \times 10} = 0.66 \text{ sec}.$$

$$5\%\ t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.6 \times 10} = 0.5 \text{ sec}.$$

(v) Response :-



3) The response of a servomechanism is $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

= Solution:-

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

Take laplace transform,

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$C(s) = \frac{(s+60)(s+10) + 0.2s(s+10) - (1.2)s(s+60)}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

Input : unit step signal, $R(s) = 1/s$.

$$C(s) = \frac{1}{s}\left(\frac{600}{s(s+60)(s+10)}\right)$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}}$$

In general, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

compare, $\omega_n^2 = 600 \Rightarrow \boxed{\omega_n = 24.49 \text{ rad/sec}}$

$$2\xi\omega_n = 70$$

$$\boxed{\xi = 1.42}$$

$\therefore$ undamped natural freq $= \omega_n = 24.49$ rad/sec.

4) The unity feedback system is characterized by an open loop transfer function $G(s) = K/s(s+10)$. Determine gain K, so that system will have damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time at peak overshoot for a unit step input.

= Solution: $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$= \frac{K/s(s+10)}{1+K/s(s+10)} = \frac{K}{s^2+10s+K}$$

by comparing with $\Rightarrow \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

we get, $\omega_n^2 = K$; $2\xi\omega_n = 10$ $\quad \xi = \frac{10}{2\omega_n} = \frac{10}{2\sqrt{K}}$

$\boxed{\omega_n = \sqrt{K}}$ $\quad \xi = 0.5 \rightarrow$ given.

$\xi = \frac{10}{2\sqrt{K}} \Rightarrow 0.5 = \frac{5}{\sqrt{K}}$

$\boxed{K = 100}$ $\quad \therefore \boxed{\omega_n = 10 \text{ rad/sec}}$

Peak overshoot :- $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.163$

$\% M_p = 16.3\%$

Peak time :-

5) The open loop transfer function of a unity feedback system is given by $G(s) = K/s(sT+1)$, where K & T are positive constant. By what factor should the amplifier gain K be reduced, so that peak overshoot of unit step response of the system is reduced from 75% to 25%.

⇒ Solution: $\frac{C(s)}{R(s)} = \frac{K/s(sT+1)}{1+K/s(sT+1)} = \frac{K}{Ts^2+s+K} = \frac{K/T}{s^2+\frac{1}{T}s+K/T}$

compare, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi_p\omega_n s+\omega_n^2} = \frac{K/T}{s^2+\frac{1}{T}s+K/T}$

on comparing, we get,

$\omega_n^2 = K/T \quad ; \quad 2\xi_p\omega_n = 1/T$

$\omega_n = \sqrt{K/T} \qquad \xi_p = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$

$$\boxed{\omega_n = \sqrt{K/T} \;;\; \xi_p = 1/2\sqrt{KT}}$$

peak overshoot M_p is reduced (75% to 25%) by increasing ξ_p. The ξ_p is increased by reducing the K.

$$\boxed{M_p = e^{-\xi_p\pi/\sqrt{1-\xi_p^2}}}$$

Taking natural logarithm on both sides,

$\ln M_p = -\xi_p\pi/\sqrt{1-\xi_p^2}$

Squaring on both sides,

$$(\ln M_p)^2 = \xi_\varphi^2 \pi^2 / 1-\xi_\varphi^2$$

$$(\ln M_p)^2 (1-\xi_\varphi^2) = \xi_\varphi^2 \pi^2$$

$$(\ln M_p)^2 - \xi_\varphi^2 (\ln M_p)^2 = \xi_\varphi^2 \pi^2$$

$$(\ln M_p)^2 = \xi_\varphi^2 (\pi^2 + (\ln M_p)^2)$$

$$\xi_\varphi^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

but $\xi_\varphi = \frac{1}{2\sqrt{KT}} \Rightarrow \boxed{\xi_\varphi^2 = \frac{1}{4KT}}$

$$\therefore \frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$\boxed{K = \frac{\pi^2 + (\ln M_p)^2}{4T(\ln M_p)^2}}$$

At $M_p = 0.75$, $K = K_1 = \dfrac{\pi^2 + (\ln 0.75)^2}{4\pi (\ln 0.75)^2}$

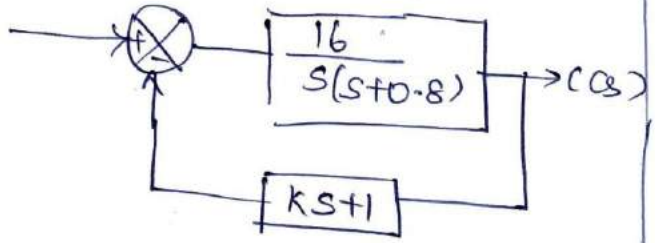
$$\boxed{K_1 = 30.1/T}$$

At $M_p = 0.25$, $K = K_2 = \dfrac{\pi^2 + (\ln 0.25)^2}{4\pi (\ln 0.25)^2}$

$$\boxed{K_2 = 1.53/T}$$

6) A position control system with velocity feedback as shown. Given that $\xi = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.

= <u>Solution</u> :-



$$T.F = \frac{C(S)}{R(S)} = \frac{16/S(S+0.8)}{1+\left(16/S(S+0.8)\right)(KS+1)}$$

$$T.F = \frac{16}{S^2+(0.8+16K)S+16}$$

on comparing we get,

$\omega_n^2 = 16$ $\qquad$ $0.8+16K = 2\xi\omega_n$.

$\boxed{\omega_n = 4\,rad/sec}$ $\qquad$ $\xi = 0.5$ (given)

$\therefore \boxed{K = 0.2}$

$$T.F = \frac{C(S)}{R(S)} = \frac{16}{S^2+4S+16}$$

(i) Damped frequency : $\omega_d = \omega_n\sqrt{1-\xi^2} = 3.464\,rad/sec$.

(ii) rise time : $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.047}{3.464} = 0.6046$.

(Assume $\theta = 1.047$) $\Rightarrow \theta = \tan^{-1}\sqrt{1-\xi^2}/\xi$

(iii) peak time : $t_p = \frac{\pi}{}$

(iv) $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.163$

$\boxed{\% M_p = 16.3\%}$

(v) settling time :-

For 2% error, $t_s = \frac{4}{\xi\omega_n} = 2 sec$

For 5% error, $t_s = \frac{3}{\xi\omega_n} = 1.5 sec$.

7) A unity feedback control system is characterized by following open loop transfer function $G(s) = (0.4s+1)/s(s+0.6)$. Determine transient response for unit step input.

= Solution :-

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{(0.4s+1)/s(s+0.6)}{1+((0.4s+1)/s(s+0.6))} = \frac{0.4s+1}{s^2+s+1}$$

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{0.4s+1}{s^2+s+1}\right)$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$0.4s+1 = A(s^2+s+1) + (Bs+C)s.$$

$\boxed{A=1}$ when $s=0$,

Equating s term, $0.4 = A + C$ $\therefore \boxed{C = -0.6}$

$$C(S) = \frac{1}{S} - \frac{S+0.6}{S^2+S+1} = \frac{1}{S} - \frac{S+0.6}{(S^2+2\times0.5S+0.5^2)+0.75}$$

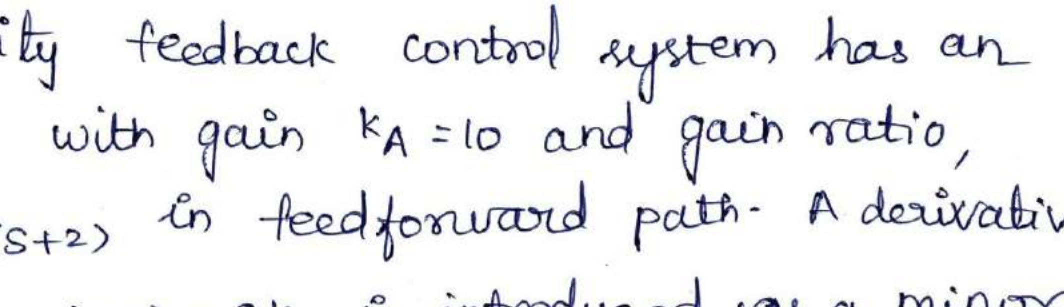
$$C(S) = \frac{1}{S} - \frac{S+0.5}{(S+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(S+0.5)^2+0.75}$$

Take inverse L.T,

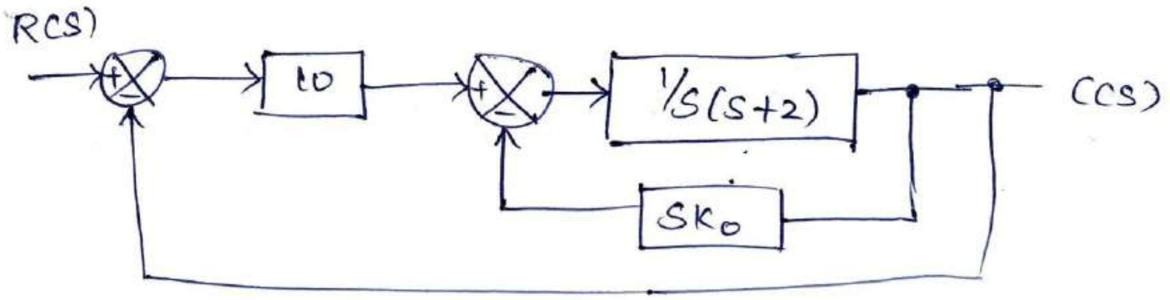
$$C(t) = 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}e^{-0.5t}\sin\sqrt{0.75}\,t$$

8. A unity feedback control system has an amplifier with gain $K_A = 10$ and gain ratio, $G(S) = 1/S(S+2)$ in feedforward path. A derivative feedback $H(S) = SK_0$ is introduced as a minor loop around $G(S)$. Determine derivative feedback constant K_0, so that the system damping factor is 0.6.

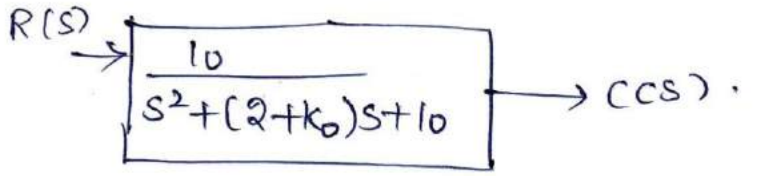
= solution:-



where $K_A = 10$, $G(S) = \frac{1}{S(S+2)}$, $H(S) = SK_0$



⇓ After simplification



$\therefore \quad \dfrac{C(S)}{R(S)} = \dfrac{10}{S^2+(2+K_0)S+10}$

on compare, $\dfrac{C(S)}{R(S)} = \dfrac{\omega_n^2}{S^2+2\xi_f\omega_n S+\omega_n^2}$

$\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} = 3.162$ rad/sec.

$2 + K_0 = 2\xi_f\omega_n$.

$K_0 = 2 \times 0.6 \times 3.162 - 2 = 1.7944$.

$\boxed{K_0 = 1.7944}$

9) A closed loop servo is represented by the differential equation $\dfrac{d^2c}{dt^2} + 8\dfrac{dc}{dt} = 64e$. where c is displacement of output shaft, r is displacement of input shaft and $e = r - c$. Determine undamped natural frequency, damping ratio, M_p for unit step

= Solution :-

The mathematical equation governing the System are,

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e.$$

put $\boxed{e = r - c}$ $\Rightarrow \frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r-c).$

Taking Laplace transform,

$$s^2C(s) + 8sC(s) = 64[R(s) - C(s)]$$

$$[s^2 + 8s + 64]C(s) = 64R(s)$$

$$\therefore \quad T.F = \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64}.$$

Now, compare with second order s/m.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\therefore$ Natural undamped frequency : $\boxed{\omega_n = 8 \text{ rad/sec}}$

Damping ratio : $\boxed{\xi = 0.5}$

$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.163.$

and it input is q $\boxed{12 \text{ units}}$ $\quad \therefore M_p = 0.163 \times 12 = 1.956$

$\% M_p = 1.956 \times 100 = 195.6\%$

UNIT: III FREQUENCY RESPONSE AND SYSTEM ANALYSIS

II yr ECE

closed loop frequency response - performance specification in frequency domain - Frequency response of standard second order system - Bode plot - polar plot - Nyquist plots - Design of compensators using Bode plots - Cascade Lead compensation - Cascade lag compensation - cascade lag lead compensation.

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INTRODUCTION

There are many time-domain methods to analyse control systems yet even after several decades, frequency response methods continue to be used for analysis and design. It should be remembered that during 1930 to 1940 when the design of several military systems were first undertaken in USA and Europe, only methods to available for analysis of stability, were those given by Bode & Nyquist.

The frequency response analysis deals with study of steady state response of the system to sinusoidal input of variable frequency.

Initially, frequency response analysis is the determination of system transfer function. Then it is expressed in terms of Magnitude & phase angle.

$M(s) = |M| \angle \phi$

↳ Phase angle

→ Magnitude.

—×——×——

Closed loop frequency Response

For a single loop control system configuration, closed loop transfer function is,

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

under sinusoidal steady state $\boxed{s=j\omega}$, then

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

Then sinusoidal steady state transfer function $M(j\omega)$ may be expressed in terms of its magnitude and phase.

$$\boxed{M(j\omega) = |M(j\omega)| \angle M(j\omega)}$$

(or)

$$\boxed{M(j\omega) = Re[M(j\omega)] + jIm[M(j\omega)]}$$

Magnitude of $M(j\omega)$ is :-

$$|M(j\omega)| = \left|\frac{G(j\omega)}{1+G(j\omega)H(j\omega)}\right| = \frac{|G(j\omega)|}{|1+G(j\omega)H(j\omega)|}$$

phase of $M(j\omega)$ is

$$\angle M(j\omega) = \phi_M(j\omega) = \angle G(j\omega) - \angle[1+G(j\omega)H(j\omega)]$$

Let us consider T.F of first order system,

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{1+s\tau} \quad (\text{or}) \quad C(s) = R(s)\left[\frac{1}{1+s\tau}\right]$$

Replace 's' by '$j\omega$',

$$G(j\omega) = \frac{1}{1+j\omega\tau}$$

Then system is subjected to sinusoidal input and therefore, $\boxed{R(s) = A_i \sin\omega t}$

$$G(j\omega) = C(j\omega) = R(j\omega)\left(\frac{1}{1+j\omega\tau}\right)$$

$$= A_i \sin\omega t \left(\frac{1}{1+j\omega\tau}\right)$$

Magnitude of output : $A_o = |C(j\omega)| = \dfrac{A_i}{\sqrt{1+\omega^2\tau^2}}$

Magnitude of input : A_i.

The dimensionless ~~of~~ ratio of output to input is given by, $M = \left|\dfrac{A_o}{A_i}\right| = \dfrac{1}{\sqrt{1+\omega^2\tau^2}}$

Phase angle : $\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau)$

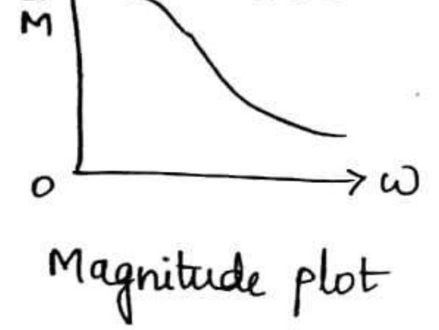
$$\text{Time lag} = \frac{1}{\omega}\tan^{-1}\omega\tau.$$

Now, let us plot M versus ω and ϕ versus ω.

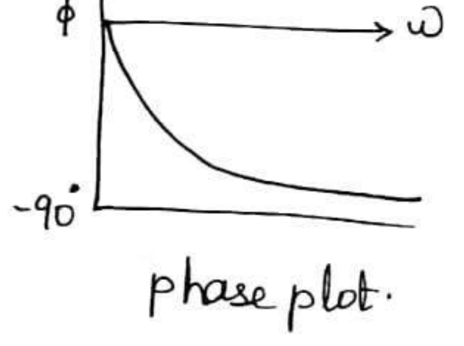
when $\omega = 0$, $M = \frac{A_o}{A_i} = \frac{1}{\sqrt{1+\omega^2\tau^2}} = 1$, $\phi = 0^\circ$.

$\omega = \infty$, $M = \frac{A_o}{A_i} = \frac{1}{\sqrt{1+\omega^2\tau^2}} = 0$, $\phi = -90^\circ$

As ω increases from 0 to ∞, magnitude gradually decreases from 1 to 0 and angle of lag ϕ increases from 0° to -90°. Thus higher the frequency, higher is the attenuation (decay) of the output & greater is angle of lag between output & input.



Magnitude plot



phase plot.

Let us consider transfer function of an system is,

$$G(s) = \frac{K}{s(s\tau+1)}$$

$$G(j\omega) = \frac{K}{j\omega(j\omega\tau+1)} \qquad (\because s = j\omega)$$

$$G(j\omega) = \frac{k}{\;} \qquad (\because j^2 = -1)$$

$$G(j\omega) = \frac{K}{j\omega - \omega^2 \tau}$$

Magnitude : $|G(j\omega)| = \dfrac{K}{\sqrt{\omega^2 + \omega^4 \tau^2}}$

Phase angle : $\angle G(j\omega) = -\tan^{-1}\left(\frac{b}{a}\right) = -\tan^{-1}\left(\frac{-\omega}{-\omega^2\tau}\right)$

$$= -\tan^{-1}\left(\frac{1}{\omega\tau}\right)$$

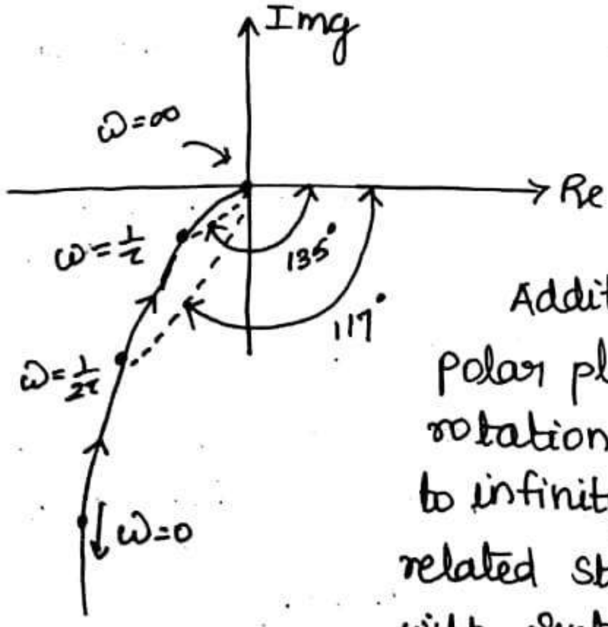
At $\omega = 0$, $|G(j\omega)| = \infty$, $\phi(j\omega) = -90^\circ$

$\omega = \frac{1}{2\tau}$, $|G(j\omega)| = \frac{4K\tau}{\sqrt{5}}$, $\phi(\omega) = -117^\circ$

$\omega = \frac{1}{\tau}$, $|G(j\omega)| = \frac{K\tau}{\sqrt{2}}$, $\phi(\omega) = -135^\circ$

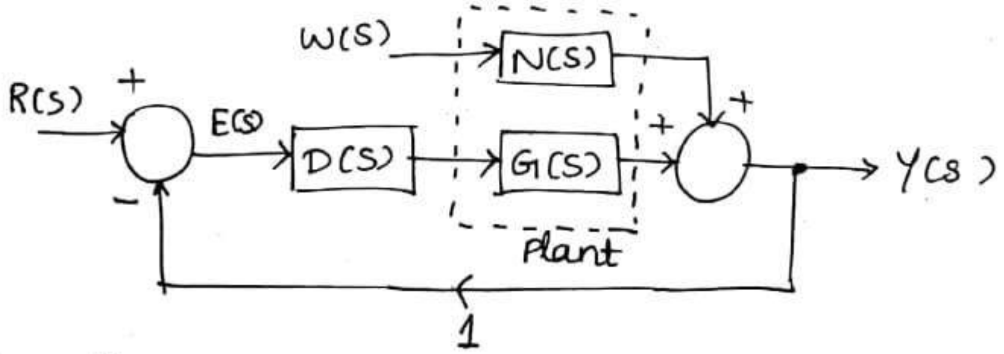
$\omega = \infty$, $|G(j\omega)| = 0$, $\phi(\omega) = -180^\circ$

The polar plot can now be drawn using above values,



Addition of a pole, the polar plot get a clockwise rotation by 90° as ω tends to infinity. Nyquist in 1932, related study of polar plot

Performance Specification in frequency domain.



In the design of linear control systems using frequency domain methods, it is necessary to define a set of specifications so that Quality of transient response can be described using frequency responses characteristics.

(i) Resonance peak: (M_r) :-

This is the maximum Value of M and is denoted as M_r. The magnitude of resonance peak M_r provide us information about the relative stability of the system. Large resonant peak corresponds to large overshoot in the transient response. $\boxed{M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}}$

(ii) Resonant frequency: (ω_r) :-

It is the frequency at which resonant peak occurs, ie., maximum

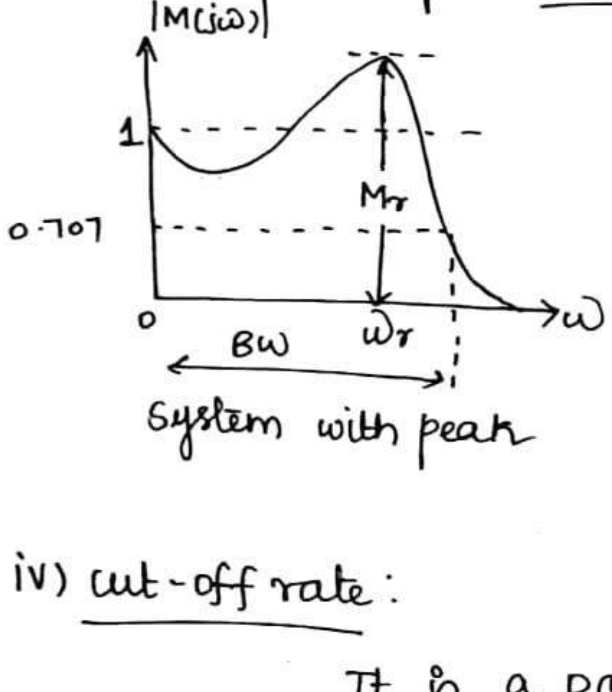
$$\boxed{\omega_r = \omega_n\sqrt{1-2\xi^2}}$$

Value of M_r occurs. High value of ω_r indicates that the time response of output is faster as peak time is inversely proportional to ω_r.

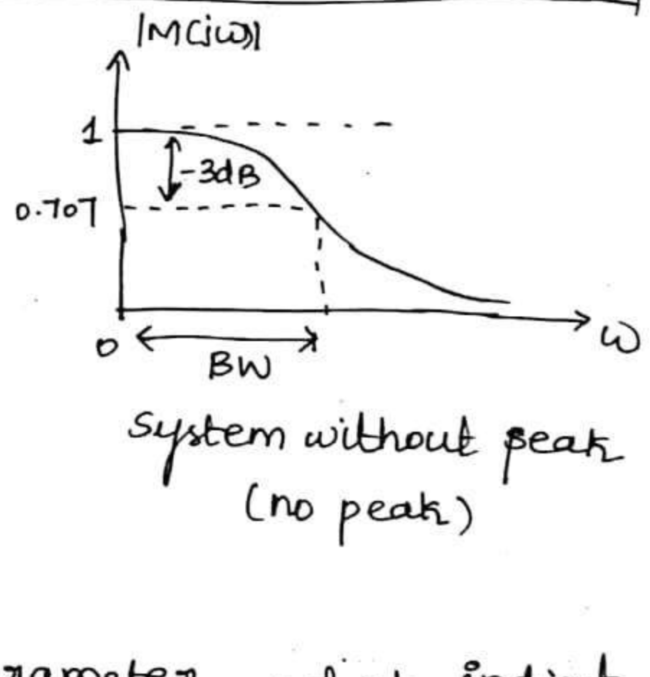
iii) Bandwidth (BW):-

It is the frequency at which magnitude M is $1/\sqrt{2} = 0.707$ or $-3dB$ times its value at zero frequency.

$$\boxed{B.W = \omega_n \sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}}$$



System with peak



System without peak (no peak)

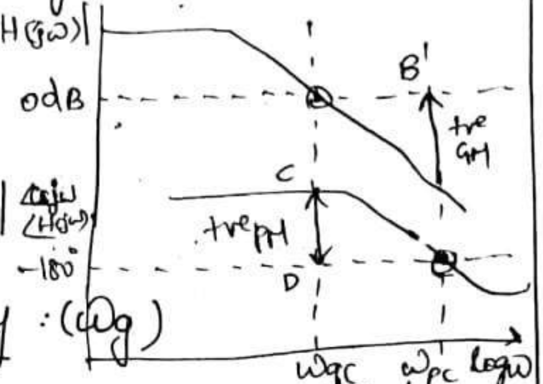
iv) cut-off rate:

It is a parameter which indicate the ability of a system in distinguishing signals from noise. It is present at slope of $|M(j\omega)|$. & Apparently, if two systems have same Bandwidth, but cutoff rates may be different.

(v) Gain Margin: (GM)

It is the factor by which gain can be increased to drive the system to the verge of instability.

$$GM = \frac{1}{|G(j\omega_\phi)\,H(j\omega_\phi)|}$$



(vi) Gain cross over frequency: (ω_g)

It is the frequency at which gain cross over the point.

$$|G(j\omega_g)\,H(j\omega_g)| = 1 \quad \text{(or)} \quad 0\,dB \quad (\because \log 1 = 0\,dB)$$

(vii) phase margin :- (PM) (or) (ϕ_M)

It is the angle by which phase $G(j\omega).H(j\omega)$ can be decreased to drive the system to verge of instability.

$$\phi_M = 180^\circ + \angle G(j\omega_g)\,H(j\omega_g).$$

(viii) phase cross over frequency: ω_ϕ

It is the frequency at which phase cross over point.

$$\angle G(j\omega_\phi)\,H(j\omega_\phi) = -180^\circ$$

—— x —— x ——

Frequency Response of Standard Second-order System.

The open loop transfer function in a standard second order system,

$$\boxed{G(s) = \omega_n^2 / s(s+2\xi\omega_n)}$$

The closed loop transfer function in a standard second order system,

$$\boxed{G(s) = \omega_n^2 / s^2+2\xi\omega_n s+\omega_n^2}$$

For the standard second order system, There is some direct and simple correlations exist between frequency domain specifications. They are as follows,

(i) Damping Ratio and phase Margin
(ii) Response speed and Gain cross over frequency.
(iii) Response speed and resonance frequency.
(iv) Damping Ratio and Resonance peak.
(V) Response speed and Bandwidth.

The correlation of higher order systems are different and more complex. Applications of correlation are developed for standard second order to other second order & higher order is based

(i) Damping Ratio and phase Margin:-

$$G(s) = \frac{\omega_n^2}{s(s+2\xi_\phi\omega_n)}$$

Put $s=j\omega \Rightarrow G(j\omega) = \frac{\omega_n^2}{-\omega^2+j2\xi_\phi\omega_n\omega}$

Magnitude $|G(j\omega)| = \frac{\omega_n^2}{|-\omega^2+j2\xi_\phi\omega_n\omega|} = 1 \longrightarrow (1)$

The frequency ω_g, satisfy the above eqn is given,

$$\omega_n^2 = \omega_g\sqrt{\omega_g^2+(2\xi_\phi\omega_n)^2}$$

(squaring and rearranging)

(or) $\boxed{\omega_g^4 + 4\xi_\phi^2\omega_n^2\omega_g^2 - \omega_n^4 = 0}$

The roots of this eqn follow by applying Quadratic formula in terms of ω_g^2.

$$\omega_g^2 = \omega_n^2\left(-2\xi_\phi^2 \pm \sqrt{4\xi_\phi^4+1}\right)$$

($a=1$
$b=4\xi_\phi^2\omega_n^2$
$c=-\omega_n^4$)

For ω_g to be real Values, positive root must be used so that,

$$\boxed{\omega_g = \omega_n\sqrt{\sqrt{4\xi_\phi^4+1}-2\xi_\phi^2}}$$

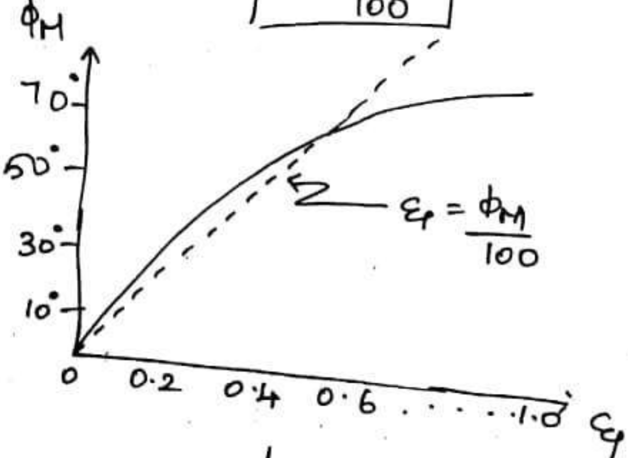
phase angle of $G(j\omega) \Rightarrow \angle G(j\omega) = -90^\circ - \tan^{-1}\left(\frac{\omega_g}{2\xi_\phi\omega_n}\right)$

$$\angle G(j\omega) = -90^\circ - \tan^{-1} \frac{\sqrt{\sqrt{4\xi_p^4+1} - 2\xi_p^2}}{2\xi_p}$$

ϕ_M (phase Margin) $= \angle G(j\omega) + 180^\circ$ $[90 - \tan^{-1}\theta = -\tan^{-1}\theta]$

$$\phi_M = 90^\circ - \tan^{-1} \frac{\sqrt{\sqrt{4\xi_p^4+1} - 2\xi_p^2}}{2\xi_p} = \tan^{-1} \frac{2\xi_p}{\sqrt{\sqrt{4\xi_p^4+1} - 2\xi_p^2}}$$

The correlation between Damping ratio and phase Margin is $\boxed{\xi_p = \frac{\phi_M}{100}}$

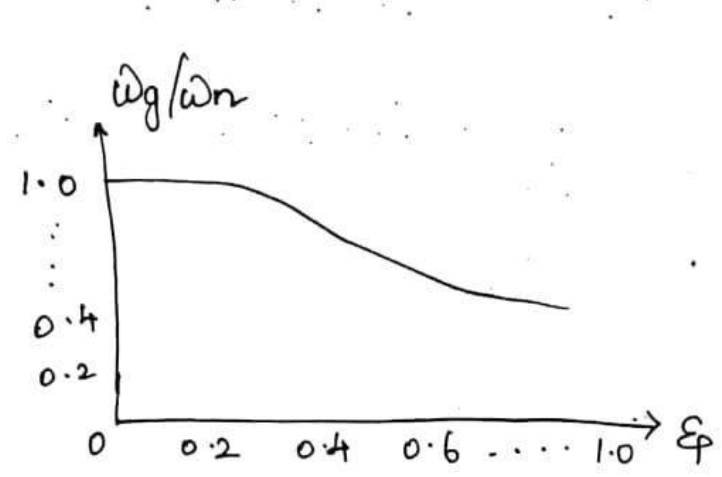


phase Margin Vs Damping Ratio.

(ii) <u>Response speed and Gain cross over frequency</u>

we know that, $\omega_g = \omega_n \sqrt{\sqrt{4\xi_p^4+1} - 2\xi_p^2}$

Frequency domain design holds the Phase margin constant while increasing the gain cross over frequency, the resulting rise time and settling time would diminish in time.



$$\boxed{\frac{\omega_g}{\omega_n} = \sqrt{\sqrt{4\xi_p^4+1} - 2\xi_p^2}}$$

(iii) Damping Ratio and Resonance peak:-

$$\text{Closed loop T.F} = M(s) = \frac{\omega_n^2}{s^2+2\xi_p\omega_n s+\omega_n^2}$$

Replace s by $j\omega$

$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi_p\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + 2j\,\xi_p\omega_n\omega + \omega_n^2} = \frac{1}{\left(1-\omega^2/\omega_n^2\right) + 2j\xi_p\left(\omega/\omega_n\right)}$$

$$M = \frac{1}{\left(1-\omega^2/\omega_n^2\right) + 2j\xi_p\left(\omega/\omega_n\right)}$$

$$|M|^2 = \frac{1}{(1-\omega^2/\omega_n^2)^2 + 4\xi_p^2(\omega^2/\omega_n^2)}$$

To find the peak value of M and frequency at which it occurs is differentiated with respect to frequency and set equal to zero,

$$\frac{dM^2}{d\omega} = 0 . \qquad \left(\frac{d}{d\omega}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right.$$

$$(u = 1,\ u' = 0)$$

$$\frac{dM^2}{d\omega} = \frac{-4(1-\omega^2/\omega_n^2)(\omega/\omega_n^2) + 8\xi_\phi^2(\omega/\omega_n^2)}{\left[(1-\omega^2/\omega_n^2)^2 + 4\xi_\phi^2(\omega^2/\omega_n^2)\right]^2}$$

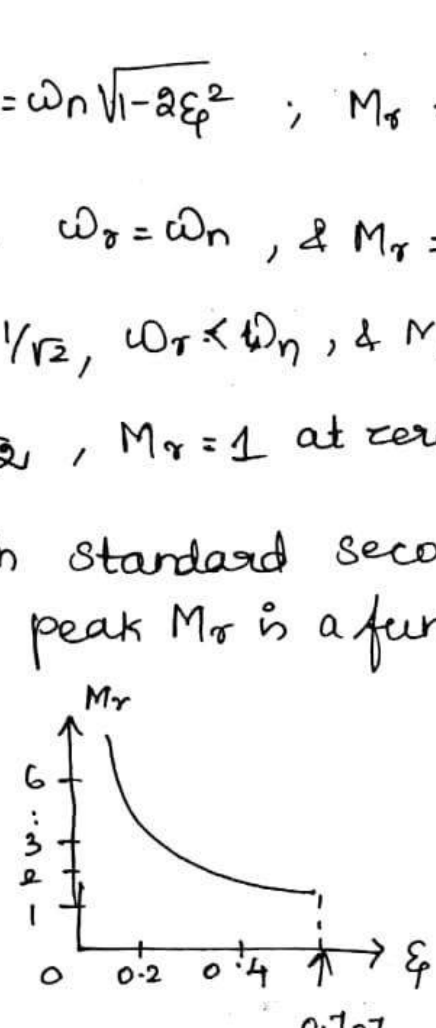
and, $\omega_r = \omega_n\sqrt{1-2\xi_\phi^2}$; $M_r = 1/2\xi_\phi\sqrt{1-\xi_\phi^2}$

If $\xi_\phi = 0$, $\omega_r = \omega_n$, & $M_r = \infty$.

$0 < \xi_\phi < 1/\sqrt{2}$, $\omega_r < \omega_n$, & $M_r > 1$

$\xi_\phi = 1/\sqrt{2}$, $M_r = 1$ at zero frequency .

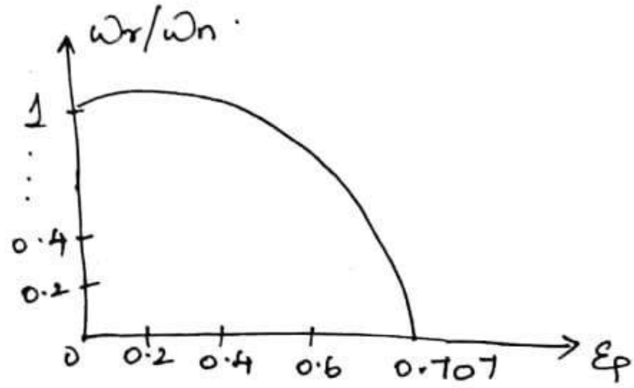
Hence in standard second order system, resonance peak M_r is a function of damping ratio only.



(iv) Response Speed and Resonant frequency .

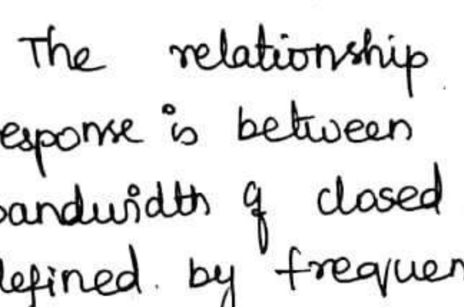
$$\boxed{\omega_r = \omega_n\sqrt{1-2\xi_\phi^2}} \quad ; \ \xi_\phi > 1/\sqrt{2} .$$

when a frequency domain design hold M_r constant while increasing the resonance frequency, resulting rise time and settling time would diminish in time domain



(V) <u>Response Speed and Bandwidth</u>:-

The relationship between frequency & time response is between speed of time response and bandwidth of closed loop frequency response is defined by frequency ω_b at which magnitude M is $1/\sqrt{2} = 0.707$ times its Value at zero freq.



The bandwidth of standard second order s/m can be found by finding that frequency for which $M = 1/\sqrt{2}$. $\Rightarrow \boxed{M^2 = 1/2}$

$$\Rightarrow \quad M^2 = \frac{1}{\left(1-\omega^2/\omega_n^2\right)^2 + 4\xi_p^2\left(\omega^2/\omega_n^2\right)} = \frac{1}{2}$$

Replace $\boxed{\omega \text{ by } \omega_b}$

$$M^2 = (1-\omega_b^2/\omega_n^2)^2 + 4\xi_p^2(\omega_b^2/\omega_n^2) = 2.$$

$$\Rightarrow (\omega_b^4/\omega_n^4) - 2(1-2\xi_p^2)(\omega_b^2/\omega_n^2) - 1 = 0.$$

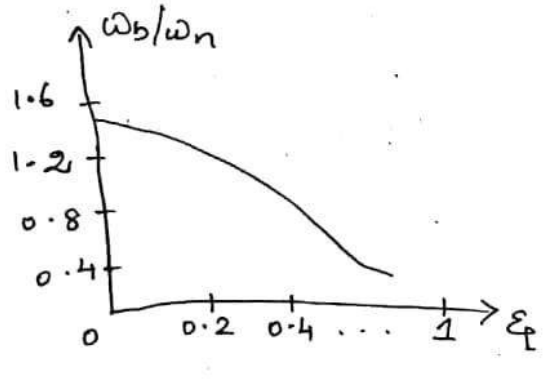
solving for ω_b^2/ω_n^2, we get,

$$(\omega_b^2/\omega_n^2) = (1-2\xi_p^2) \pm \sqrt{4\xi_p^4 - 4\xi_p^2 + 2}$$

In last equation, plus sign should be chosen, since (ω_b/ω_n) must be a positive real quantity for any ξ_p.

$$\therefore B.W = \boxed{\omega_b = \omega_n\sqrt{\left[(1-2\xi_p^2) + \sqrt{4\xi_p^4 - 4\xi_p^2 + 2}\right]}}$$

$\therefore \omega_b$ is proportional to ω_n and measure of speed of response. Raising ω_b reduces settling time and rise time of step response.



— x — x —

1. The forward path transfer function of a unity feedback control system is given as

$$G(s) = 64/s(s+5)$$

calculate resonant peak, resonant frequency and Bandwidth of closed loop system.

= Solution :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)} \quad (\because H(s)=1)$$

$$= \frac{64/s(s+5)}{1+64/s(s+5)} = \frac{64}{s^2+5s+64}$$

compare with general second order transfer fn,

$\omega_n^2 = 64 \Rightarrow \boxed{\omega_n = 8 \text{ rad/sec}}$

$2\xi\omega_n = 5 \Rightarrow \boxed{\xi = 0.3125}$

(i) Resonant frequency: $\omega_r = \omega_n\sqrt{1-2\xi^2}$

$$= 8\sqrt{1-2(0.3125)^2}$$

$$\boxed{\omega_r = 7.1 \text{ rad/sec}}$$

(ii) Resonant peak : $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2(0.3125)\sqrt{1-0.3125^2}}$

$$\boxed{M_r = 1.78}$$

(iii) Bandwidth: $\omega_b = \omega_n\sqrt{1-2\xi^2+(2-4\xi^2+4\xi^4)^{1/2}}$

$$\boxed{\omega_b = 11.25 \text{ rad/sec}}$$

2. Find the Bandwidth of the system whose transfer function is $1/s+1$.

= Solution: $T(s) = \frac{1}{s+1}$

$$T(j\omega) = \frac{1}{j\omega+1}$$

$$T(j\omega) = \frac{1}{1+j\omega}$$

$$M = \frac{1}{\sqrt{1+\omega^2}}$$

In dB, $M = 20 \log\left(\frac{1}{\sqrt{1+\omega^2}}\right) = 20 \log\left(\sqrt{1+\omega^2}\right)^{-1}$

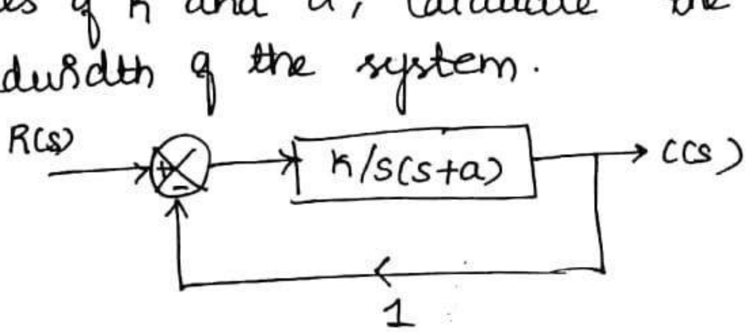
$$= -20 \log\left(\sqrt{1+\omega^2}\right)$$

For $\omega = 0$, $M = 0\,dB$.

$\omega = \omega_b$, $M = -3\,dB \Rightarrow -3 = -20 \log \sqrt{1+\omega_b}$

$$\boxed{\omega_b = 0.9976 \text{ rad/sec}}$$

3. For the given system, determine the value of k and a to satisfy the following frequency domain specifications: $M_r = 1.04$, $\omega_r = 11.55$ rad/sec. For the values of K and a, calculate the settling time and bandwidth of the system.



Solution :-

$$G(s) = \frac{K}{s(s+a)}, \quad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K/s(s+a)}{1+K/s(s+a)} = \frac{K}{s^2+as+K}$$

compare with second order T.F.

$$\omega_n^2 = K \Rightarrow \boxed{\omega_n = \sqrt{K} \text{ rad/sec}}$$

$2\xi_p\omega_n = a \Rightarrow \boxed{\xi_p = a/2\sqrt{K}}$

Given: $M_r = 1.04$, $\omega_r = 11.55$ rad/sec.

$M_r = \frac{1}{2\xi_p\sqrt{1-\xi_p^2}}$ squarring on both sides.

$\xi_p^2(1-\xi_p^2) = \frac{1}{2(1.04)^2}$

$\xi_p^4 - \xi_p^2 - 0.23114 = 0 \Rightarrow \boxed{\xi_p^2 = 0.6373, 0.3676}$ roots.

$\therefore \boxed{\xi_p = 0.6021}$ For $\xi_p > 0.707$ for which M_r does not exist.

$\omega_r = \omega_n\sqrt{1-2\xi_p^2}$ $\quad$ ($\because$ $\omega_r = 11.55$ rad/sec

$\xi_p^2 = 0.3676$.

$\boxed{\omega_n = 22.033 \text{ rad/sec}}$

$\therefore$

$\boxed{K = \omega_n^2 = 485.453}$

$\xi_p = \frac{a}{2\sqrt{K}}$

$a = 2 \times 0.6021 \times 22.033$

$\boxed{a = 26.5321}$

$\therefore$ $K = 485.453$; $a = 26.5321$.

(ii) $T_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6021 \times 22.033} = 0.3015$ sec.

$$B\omega = \omega_n \sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}} = 25.237 \text{ rad/sec.}$$

—— x —— x ——

BODE PLOT

Frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are,

(i) Bode plot (or) logarithmic plot
(ii) polar plot (or) Nyquist plot
(iii) Nichols plot
} ⇒ drawn for open loop systems.

(iv) M and N circles
(v) Nichols chart.
} ⇒ drawn to determine frequency response of unity feedback closed loop systems.

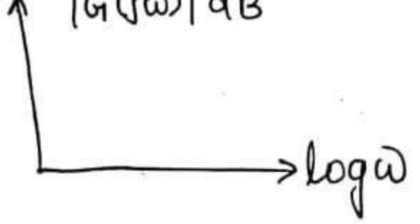
Bode plot :-

It is named after Hendricks. W. Bode. Bode plot is one of the powerful graphical methods of analyzing and designing control systems. It consists of two graphs.

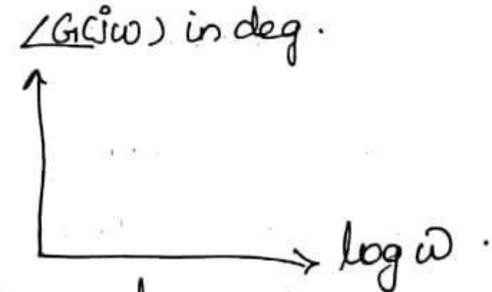
(i) plot logarithm of magnitude of sinusoidal transfer function Vs frequency in logarithmic scale.

(ii) plot phase angles in degrees Vs logarithmic frequency.

The main advantage of using logarithmic plot is that multiplication of magnitudes can be converted into addition.



Magnitude plot



phase plot.

It can be plotted in semilog graph paper. In such paper, x-axis is divided into logarithmic scale which is non linear one. while Y-axis is divided in to linear scale. hence it is called semi log paper.

In x-axis distance between 1 & 2 is greater than 2 & 3 and so on. similarly, on such scale distance between 1 and 10 is equal to distance between 10 & 100. or between 100 & 1000 and so on. This distance is called 1 decade.

$\because \log 1 = 0, \log 10 = 1, \log 100 = 2$. & so on.

The main advantage of the scale is wide range of frequencies can be accomodated on a single graph paper.

1. Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec.

$$G(s) = Ks^2/(1+0.2s)(1+0.02s)$$

= Solution:

$$T.F = G(s) = \frac{Ks^2}{(1+0.2s)(1+0.02s)}$$

Replace 's' by 'jω'

$$T.F = G(j\omega) = \frac{K(j\omega)^2}{(1+0.2j\omega)(1+0.02j\omega)}$$

let K = 1, $\therefore$ $\boxed{G(j\omega) = \frac{(j\omega)^2}{(1+j0.2\omega)(1+j0.02\omega)}}$

Step 1:- Finding out corner frequencies.

$$\text{corner freq} = \frac{\text{Real value}}{\text{Imaginary value}}$$

$$\therefore \omega_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$$

$$\omega_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec}.$$

Step 2:- Finding out change in slope at corner frequency.

Sno.	Term	corner freq rad/sec	slope dB/dec	change in slope (dB/dec)
1	$(j\omega)^2$	-	+40	
2.	$\frac{1}{1+j0.2\omega}$	$\omega_{c1} = 5$	-20	$40-20=20$
3.	$\frac{1}{1+j0.02\omega}$	$\omega_{c2} = 50$	-20	$20-20=0$

step 3:- choosing low frequency (ω_L) & high frequency (ω_h).

choose a low frequency ω_L such that $\boxed{\omega_L < \omega_{c1}}$
and choose high frequency ω_H such that $\boxed{\omega_h > \omega_{c2}}$

Assume, $\omega_l = 0.5$ rad/sec & $\omega_h = 100$ rad/sec.

let $A = |G(j\omega)|$ in dB.

At $\omega = \omega_l = 0.5 \Rightarrow A = 20 \log \underbrace{|(j\omega)^2|}_{\text{Real value in Numerator part of } G(s)} = 20\log(\omega^2) = -12$ dB

At $\omega = \omega_{c1} = 5 \Rightarrow A = 20 \log |(j\omega)^2| = 20 \log(\omega^2) = 28$ dB.

At $\omega = \omega_{c2} \Rightarrow A = \left[(\text{slope from } \omega_{c1} \text{ to } \omega_{c2}) \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A\Big|_{\omega=\omega_{c1}}$

$= 20 \times \log \frac{50}{5} + 28 = 48$ dB

At $\omega = \omega_h \Rightarrow A = \left[(\text{slope from } \omega_{c2} \text{ to } \omega_h) \times \log \frac{\omega_h}{\omega_{c2}}\right] + A_{(\text{at } \omega = \omega_{c2})}$

$$= 0 \times \log \frac{100}{50} + 48 = 48 \text{ dB}.$$

magnitude values		points in graph
$\therefore\ \omega_l = -12$	at 0.5 rad/sec $\Rightarrow$	a
$\omega_{c1} = 28$	at 5 rad/sec $\Rightarrow$	b
$\omega_{c2} = 48$	at 50 rad/sec $\Rightarrow$	c
$\omega_h = 48$	at 100 rad/sec $\Rightarrow$	d

Join the points by straight lines in semilog graph paper and mark the slope on respective region.

step 4:- calculate phase angle

$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}\left(\frac{0.2\omega}{1}\right) - \tan^{-1}\left(\frac{0.02\omega}{1}\right)$$

Sno	ω rad/sec $(\omega \times \frac{180}{\pi})^\circ$ deg	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\phi = \angle G(j\omega)$ deg $\simeq$	Points in graph
1	0.5	5.7	0.6	174	e
2	1	11.3	1.1	168	f
3	5	45	5.7	130	g
4	10	63.4	11.3	106	h
5	50	84.3	45	50	i
6	100	87.1	63.4	30	j

Plot the phase points in semilog sheet and join the points by smooth curve..

Step 5 :- calculation of K.

Given that gain cross over frequency is 5 rad/sec. At $\omega = 5$ rad/sec, gain is 28 dB. Hence to every point of magnitude plot a dB gain of -28 dB should be added. The addition of -28 dB shifts plot downwards

$$\therefore 20 \log K = -28 \text{ dB}$$

$$\log K = \frac{-28}{20}$$

$\boxed{K = 0.0398}$ ⇒ corresponding magnitude plot is plotted in semilog graph.

Note: The frequency $\omega = 5$ rad/sec is a corner frequency. Hence in exact plot, the dB gain at $\omega = 5$ rad/sec will be 3 dB less than actual plot.

$$\therefore 20 \log K = -25 \text{ dB}$$

$$\boxed{K = 0.0562}$$

Bode plot is sketched in semilog graph paper.

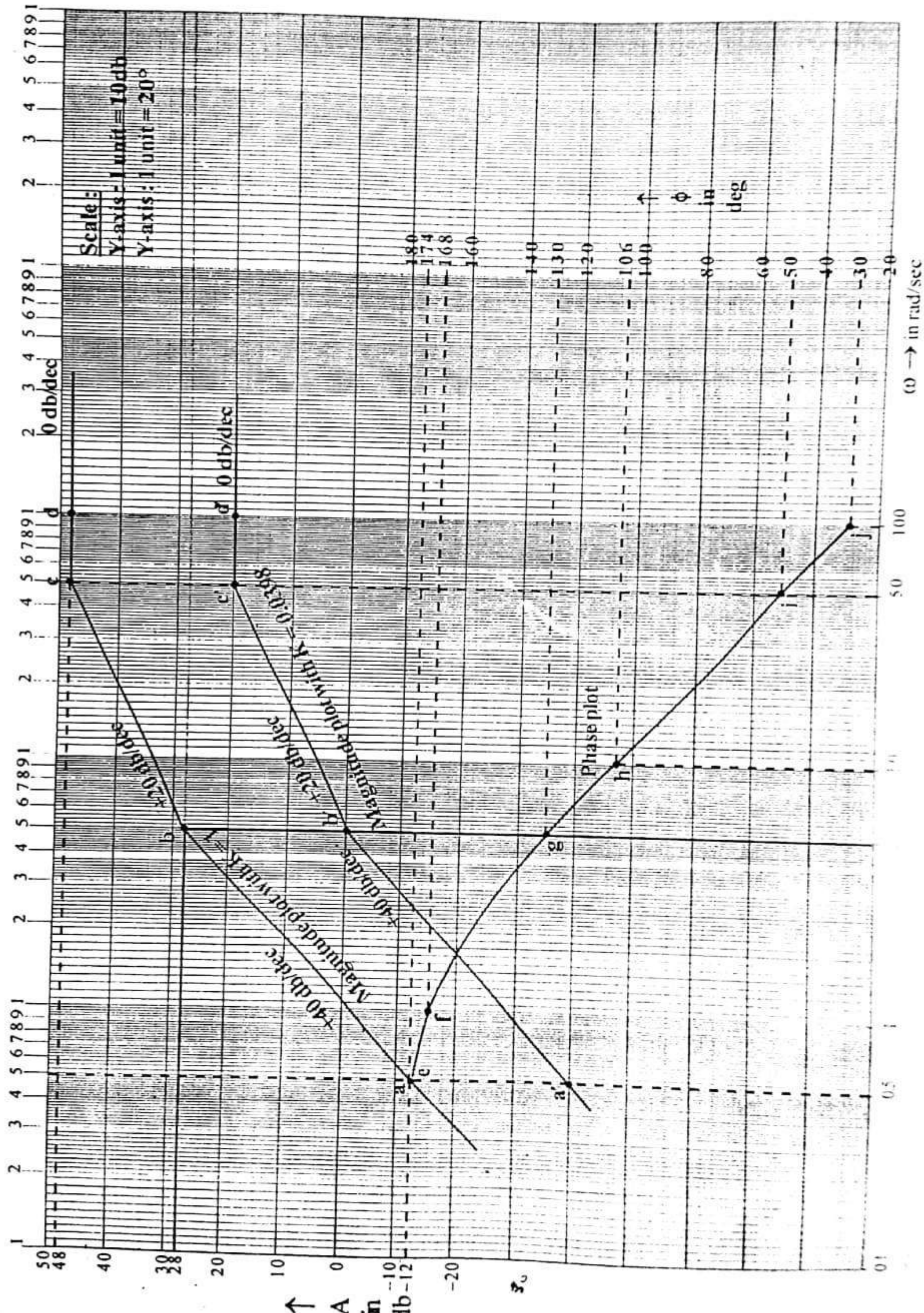


Fig 4.1.1.: Bode plot of transfer function, $G(j\omega) = \dfrac{K(j\omega)^2}{(1 + j0.2\omega)(1 + j0.02\omega)}$

2. Sketch Bode plot for following transfer function and determine phase margin and gain margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

= solution :-

on comparing $G(s)$ with second order T.F,

$\omega_n^2 = 100 \Rightarrow \boxed{\omega_n = 10\,rad/sec}$; $2\xi\omega_n = 16 \Rightarrow \boxed{\xi = 0.8}$

let convert s domain to Bode form.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{0.75(1+0.2s)}{s(1+0.16s+0.01s^2)}$$

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega(1+0.16j\omega+0.01(j\omega)^2)} = \frac{0.75(1+j0.2\omega)}{j\omega(1-0.01\omega^2+j0.16\omega)}$$

Step 1 :- Magnitude plot,

corner frequencies, $\omega_{c1} = \frac{1}{0.2} = 5\,rad/sec$

$\omega_{c2} = \omega_n = 10\,rad/sec$

because, for Quadratic factor, corner freq $= \omega_n$.

Sno	Term	corner freq rad/sec	slope dB/dec	change in slope dB/dec
1	$0.75/j\omega$	—	-20	
2.	$1+j0.2\omega$	$\omega_{c1} = 5$	20	$-20+20=0$
3.	$\frac{1}{\ldots}$	$\omega_{c2} = \omega_n = 10$	-40	$0-40=-40$

Step 2:-

$\omega_l = 0.5$ rad/sec $\quad \omega_h = 20$ rad/sec.

At $\omega = \omega_l \Rightarrow A = 20 \log \left| \frac{0.75}{\omega} \right| = 3.5\,dB$

$\omega = \omega_{c1} \Rightarrow A = 20 \log \left| \frac{0.75}{\omega} \right| = -16.5\,dB$

$\omega = \omega_{c2} \Rightarrow A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{(at\, \omega = \omega_{c1})}$

$= 0 \times \log \frac{10}{5} + (-16.5) = -16.5\,dB$.

$\omega = \omega_h \Rightarrow A = \left[(\text{slope from } \omega_{c2} \text{ to } \omega_h) \times \log \frac{\omega_h}{\omega_{c2}}\right] + A_{at\, \omega = \omega_{c2}}$

$= -40 \times \log \frac{20}{10} + (-16.5) = -28.5\,dB$.

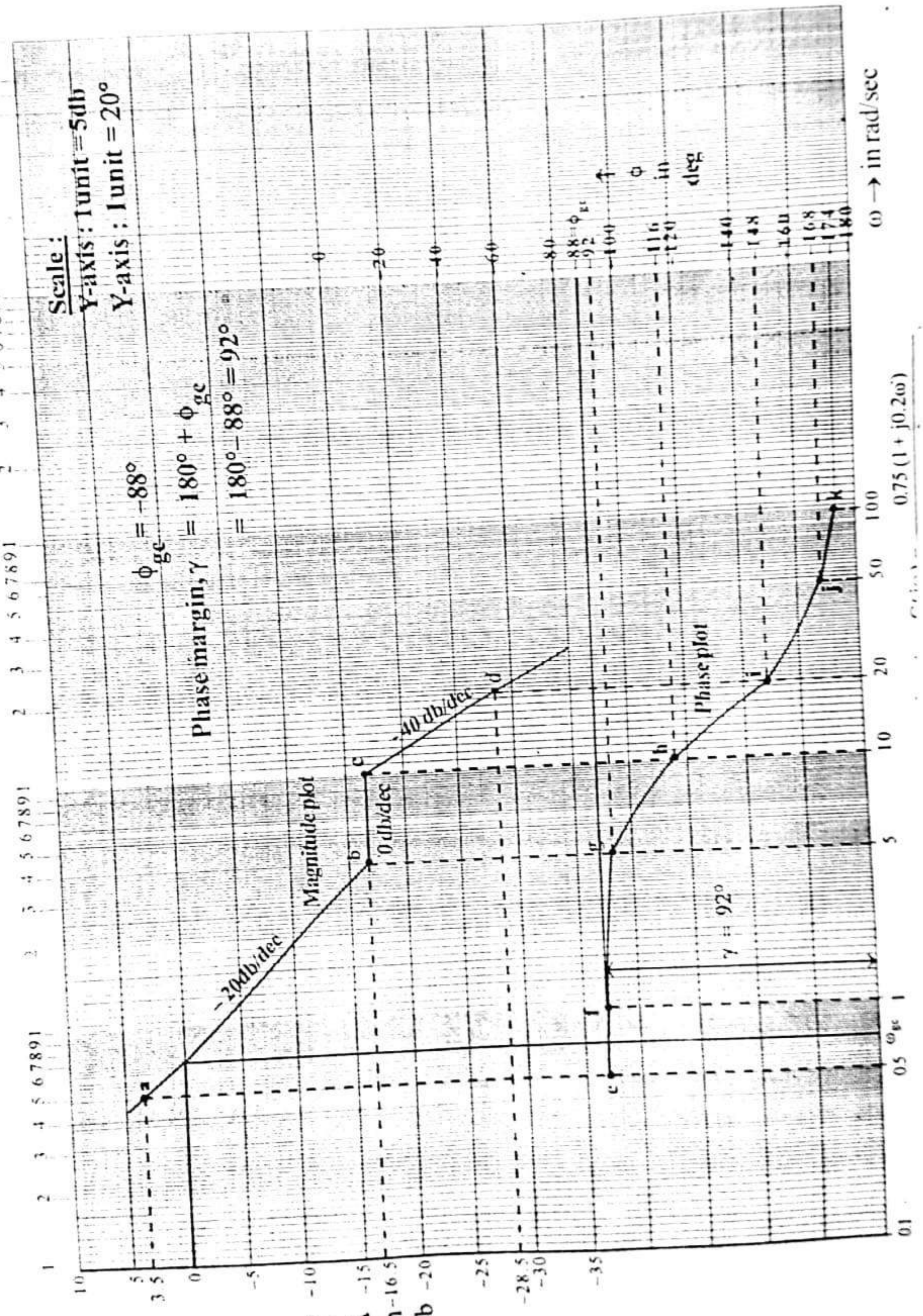
Sno	ω	ω-Value	magnitude values [dB]	points in graph.
1.	ω_l	0.5	3.5	a
2	ω_{c1}	5	−16.5	b
3	ω_{c2}	10	−16.5	c
4	ω_h	20	−28.5	d

Step 3:- phase plot :-

$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90° - \tan^{-1} \frac{0.16\omega}{1 - 0.01\omega^2}$ for $\omega \leq \omega_n$

$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90° - \tan^{-1} \left(\frac{0.16\omega}{1 - 0.01\omega^2} + 180°\right)$ for $\omega > \omega_n$

Note:- In Quadratic factors, the phase varies from 0° to 180°. but Calculator calculates $\tan^{-1}$ only between 0° to 90°. Hence a correction of 180° should be added alway...


Scale :
X-axis : 1 unit = 5db
Y-axis : 1 unit = 20°
$\phi_{gc} = -88°$
Phase margin, $\gamma = 180° + \phi_{gc}$
$= 180° - 88° = 92°$
Magnitude plot
–20db/dec
0 db/dec
–40 db/dec
Phase plot
$\gamma = 92°$
ω_{gc}
$\omega \rightarrow$ in rad/sec
ϕ in deg
$0.75(1 + j0.2\omega)$

S no	ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} \frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$	Points
1	0.5	5.7	4.6	-88	e
2	1	11.3	9.2	-88	f
3	5	45	46.8	-92	g
4	10	63.4	90	-116	h
5	20	75.9	$-46.8° + 180° = 133.2$	-148	i
6	50	84.3	$-18.4° + 180° = 161.6$	-168	j
7	100	87.1	$-9.2° + 180° = 170.8°$	-174	k

Let ϕ_{gc} be phase of $G(j\omega)$ at gain cross over frequency, ω_{gc}. from graph we get,

$$\phi_{gc} = 88°$$

$\therefore$ Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 88° = 92°$

The phase plot crosses $-180°$ only at infinity.

$|G(j\omega)|$ at infinity $= -\infty$ dB. hence gain margin $= +\infty$.

———— × ————

3. Given $G(s) = \frac{Ke^{-0.2s}}{s(s+2)(s+8)}$. Find K, so that system is stable with a) gain margin equal to 2dB.
b) phase margin equal to 45°.

= <u>Solution</u>:- let $K = 1$,

$$G(s) = \frac{e^{-0.2s}}{s(s+2)(s+8)} = \frac{0.0625\, e^{-0.2s}}{s(1+0.5s)(1+0.125s)}$$

$$G(j\omega) = \frac{0.0625\, e^{j0.2\omega}}{j\omega(1+j0.5\omega)(1+j0.125\omega)}$$

<u>Note</u>:- $|0.0625\, e^{-j0.2\omega}| = 0.0625$

$\angle 0.0625\, e^{-j0.2\omega} = -0.2\omega$ radians.

<u>Step 1</u>: Magnitude plot.

corner frequencies, $\omega_{c1} = \frac{1}{0.5} = 2$ rad/sec

$\omega_{c2} = \frac{1}{0.125} = 8$ rad/sec.

Sno	Term	corner freq rad/sec	Slope dB/dec	change in slope dB/dec
1	$\frac{0.0625}{j\omega}$	–	-20	
2.	$\frac{1}{1+j0.5\omega}$	$\omega_{c1} = 2$	-20	$-20-20=-40$
3.	$\frac{1}{1+j0.125\omega}$	$\omega_{c2} = 8$	-20	$-40-20=-60$

$\omega_l = 0.5$ rad/sec $\qquad \omega_h = 50$ rad/sec.

At $\omega = \omega_l \Rightarrow A = 20\log\left|\frac{0.0625}{j\omega}\right| = -18\,dB$

$\omega = \omega_{c1} \Rightarrow A = 20\log\left|\frac{0.0625}{j\omega}\right| = -30\,dB$

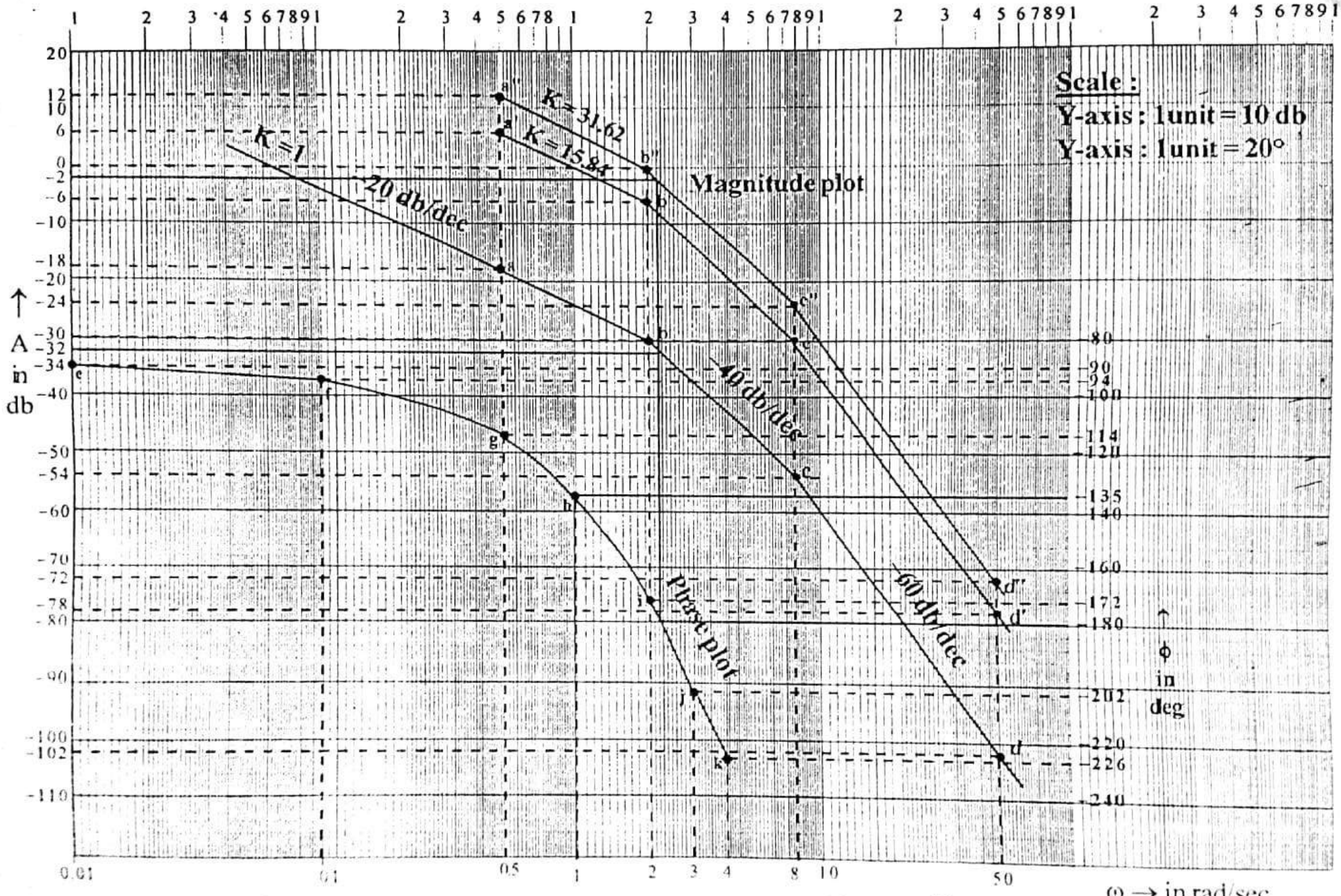


Fig 4.3.1: *Bode plot of transfer function,* $G(j\omega) = \dfrac{0.0625\,K\,e^{-j0.2\omega}}{j\omega\,(1 + j0.5\omega)\,(1 + j0.125\omega)}$

At $\omega = \omega_{c2}$,

$$A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}}\right] + A_{at(\omega=\omega_{c1})}$$

$$= -40 \times \log \frac{8}{2} + (-30) = -54\,dB.$$

At $\omega = \omega_h$,

$$A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}}\right] + A_{at(\omega=\omega_{c2})}$$

$$= -60 \times \log \frac{50}{8} + (-54) = -102\,dB.$$

sno	ω	ω value	magnitude (dB)	points
1	ω_l	0.5	−18	a
2	ω_{c1}	2	−30	b
3	ω_{c2}	8	−54	c
4	ω_h	50	−102	d

step 2:- phase plot

$$\phi = -0.2\omega \times \frac{180^\circ}{\pi} - 90^\circ - \tan^{-1} 0.5\omega - \tan^{-1} 0.125\omega$$

ω	$-0.2\omega \times \frac{180^\circ}{\pi}$	$\tan^{-1} 0.5\omega$	$\tan^{-1} 0.125\omega$	ϕ $\approx$ deg	Points
0.01	−0.1145	0.2864	0.0716	−90	e
0.1	−1.145	2.862	0.716	−94	f
0.5	−5.7	14	3.6	−114	g
1	−11.4	26	7.12	−134	h
2	−22.9	45	14	−172	i
3	−34.37	56.30	20.56	−202	j
			[illegible]	−226	k

Calculation of K:-

Phase margin $\gamma = 180^\circ + \phi_{gc}$. ($\gamma = 45^\circ$ given)

$$\phi_{gc} = \gamma - 180^\circ = -135^\circ$$

with K = 1, dB gain at $\phi = -135^\circ$ is -24 dB. This gain should be made zero to have PM of 45°. Hence to every point of magnitude plot dB gain of 24 dB should be added.

$$20 \log K = 24 \Rightarrow \boxed{K = 15.84}$$

with K = 1, gain margin $= -(-32) = 32$ dB. but required gain margin = 2 dB. Hence to every point of magnitude plot, a dB gain of 30 dB should be added. Hence the addition of 30 dB shift the plot upwards.

$$\therefore 20 \log K = 30 \Rightarrow \boxed{K = 31.62}$$

The magnitude plot with K = 15.84 & 31.62 is shown in semilog graph paper.

— × — × —

4. plot the Bode diagram for the following transfor function and obtain gain and phase cross over frequencies.

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$$

⇒ solution:-

$$G(j\omega) = \frac{10}{j\omega(1+j0.4\omega)(1+j0.1\omega)}$$

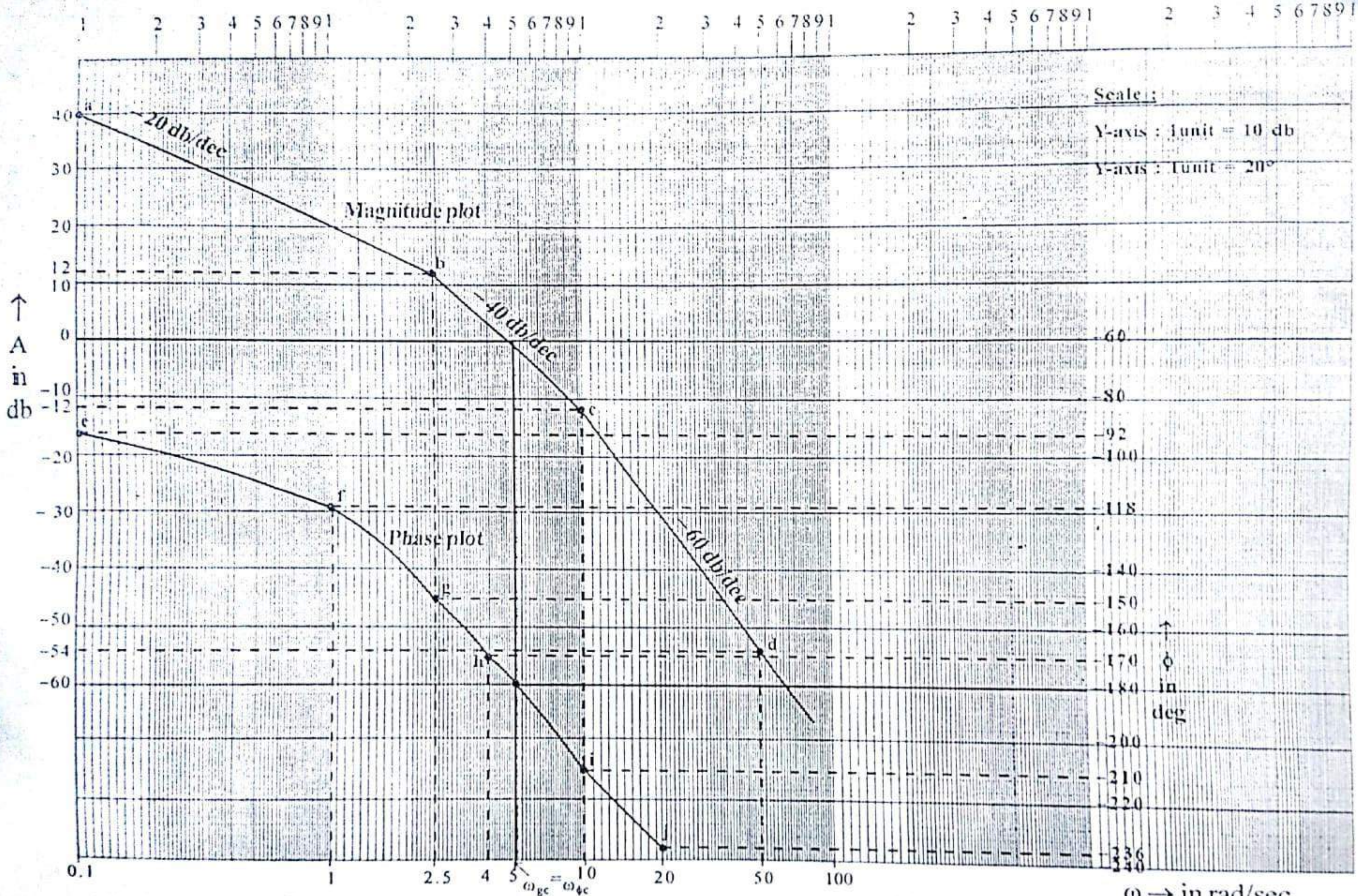


Fig 4.4.1: Bode plot of transfer function, $G(j\omega) = \dfrac{10}{j\omega(1+j0.4\omega)(1+j0.1\omega)}$.

Step 1: Magnitude plot,

$\omega_{c1} = 2.5$ rad/sec; $\omega_{c2} = \frac{1}{0.1} = 10$ rad/sec.

sno	term	corner freq	slope dB/dec	change in slope
1.	$10/j\omega$	-	-20	
2.	$1/1+j0.4\omega$	$\omega_{c1} = 2.5$	-20	$-20+20=-40$
3.	$1/1+j0.1\omega$	$\omega_{c2} = 10$	-20	$-40-20=-60$

$\omega_l = 0.1$ rad/sec, $\omega_h = 50$ rad/sec.

At $\omega = \omega_l \Rightarrow A = 20\log\left|\frac{10}{j\omega}\right| = 40\,dB$.

$\omega = \omega_{c1} \Rightarrow A = 20\log\left|\frac{10}{j\omega}\right| = 12\,dB$.

$\omega = \omega_{c2} \Rightarrow A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A \text{ at } \omega = \omega_{c1}$

$= -40 \times \log\frac{10}{2.5} + 12 = -12\,dB$.

$\omega = \omega_h, \Rightarrow A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c2}}\right] + A_{(\omega=\omega_{c2})}$

$= -60 \times \log\frac{50}{10} + (-12) = -54\,dB$.

sno	ω	ω-value	mag (dB)	points
1	ω_l	0.1	40	a
2	ω_{c1}	2.5	12	b
3	ω_{c2}	10	-12	c
		50	-54	d

Phase plot:-

$$\phi = -90° - \tan^{-1} 0.4\omega - \tan^{-1} 0.1\omega$$

Sno	ω	$\tan^{-1} 0.4\omega$	$\tan^{-1} 0.1\omega$	$\phi = \angle G(j\omega)$	Points
1.	0.1	2.29	0.57	-92	e
2	1	21.80	5.71	-118	f
3	2.5	45.0	14.2	-150	g
4	4	57.99	21.8	-170	h
5	10	75.96	45.0	-210	i
6	20	82.87	63.43	-236	j

From the graph, gain & phase cross over frequencies are found to be 5 rad/sec.

5. For the following transfer function draw bode plot and obtain gain cross-over frequency.

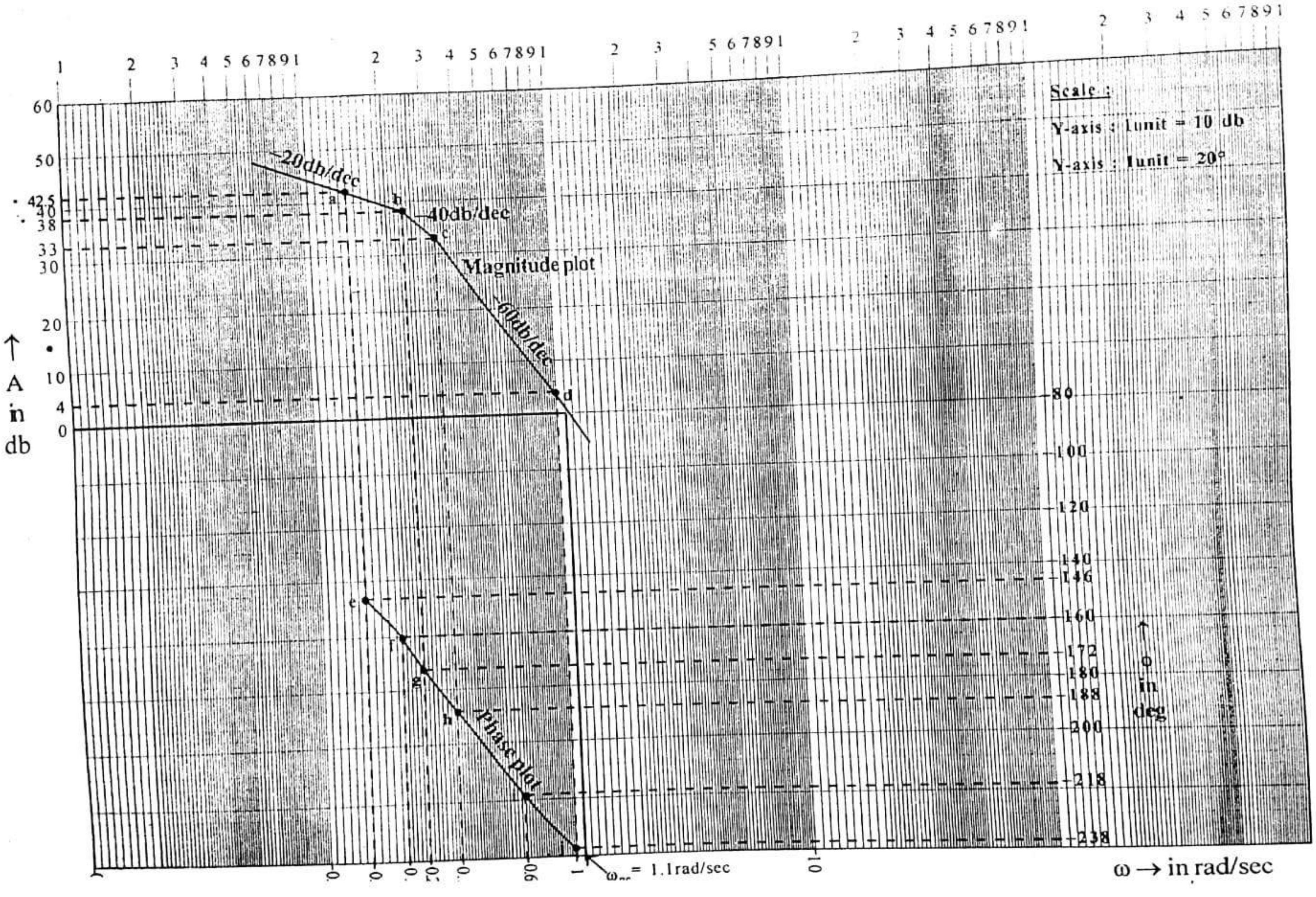
$$G(s) = 20 / s(1+3s)(1+4s)$$

= Solution:-

$$G(j\omega) = \frac{20}{j\omega(1+j3\omega)(1+j4\omega)}$$

$\omega_{c1} = \frac{1}{4} = 0.25$ rad/sec $\quad \omega_{c2} = \frac{1}{3} = 0.333$ rad/sec.

Term	corner freq	slope dB/dec	change in slope dB/dec
$20/j\omega$	—	-20	
$1/1+j4\omega$	$\omega_{c1} = 0.25$	-20	$-20-20=-40$
	$\omega_{c2} = 0.33$	-20	$-40-20=-60$


Scale :
Y-axis : 1unit = 10 db
Y-axis : 1unit = 20°
−20db/dec
−40db/dec
Magnitude plot
−60db/dec
Phase plot
A in db
φ in deg
ω → in rad/sec
ω_{gc} = 1.1rad/sec
60
50
42.5
40
38
33
30
20
10
4
0
−80
−100
−120
−140
−146
−160
−172
−180
−188
−200
−218
−238
a
b
c
d
e
f
g
h
0.6
1
10

$\omega_l = 0.15$ rad/sec , $\omega_h = 1$ rad/sec.

At $\omega = \omega_l \Rightarrow A = 20 \log \left| \frac{20}{0.15} \right| = 42.5\,dB$

$\omega = \omega_{c1} \Rightarrow A = 20 \log \left| \frac{20}{0.25} \right| = 38\,dB$.

$\omega = \omega_{c2} \Rightarrow A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log \frac{\omega_{c2}}{\omega_{c1}} \right] + A_{at\ \omega=\omega_{c1}}$

$= -40 \times \log \frac{0.33}{0.25} + 38 = 33\,dB$.

$\omega = \omega_h \Rightarrow A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c2}} \right] + A_{at\ \omega=\omega_{c2}}$

$= -60 \times \log \left(\frac{1}{0.33} \right) + 33 = 4\,dB$.

Sno	ω	ω-Values	magnitude	points
1.	ω_l	0.15	42.5	a
2.	ω_{c1}	0.25	38	b
3.	ω_{c2}	0.33	33	c
4	ω_h	1	4	d.

phase plot :- $\phi = -90° - \tan^{-1} 3\omega - \tan^{-1} 4\omega$

ω	$\tan^{-1} 3\omega$	$\tan^{-1} 4\omega$	ϕ	Points
0.15	24.22	30.96	-146	e
0.2	30.96	38.66	-160	f
0.25	36.86	45.0	-172	g
0.33	44.7	52.8	-188	h
0.6	60.14	67.38	-218	i
	[illegible]	75.96	-238	j

6: For the function $G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25s)}$, draw the Bode plot.

⇒ Solution:-

$$G(j\omega) = \frac{5(1+j2\omega)}{(1+j4\omega)(1+j0.25\omega)}$$

corner freq :- $\omega_{c1} = \frac{1}{4} = 0.25$ rad/sec

$\omega_{c2} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c3} = 1/0.25 = 4$ rad/sec.

$\omega_l < \omega_{c1}$ & $\omega_h > \omega_{c2}$ & ω_{c3} ⇒ $\omega_l = 0.1$ rad/sec

$\omega_h = 10$ rad/sec.

Term	corner freq rad/sec	slope dB/dec	change in slope dB/dec
5	—	0	
$\frac{1}{1+j4\omega}$	$\omega_{c1} = 0.25$	-20	$0-20=-20$
$1+j2\omega$	$\omega_{c2} = 0.5$	$+20$	$-20+20=0$
$\frac{1}{1+j0.25\omega}$	$\omega_{c3} = 4$	-20	$0-20=-20$.

At $\omega = \omega_l \Rightarrow A = 20\log 5 = +14$ dB

$\omega = \omega_{c1} \Rightarrow A = 20\log 5 = +14$ dB.

$\omega = \omega_{c2} \Rightarrow A = \left[\text{slope from } \omega_{c1} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c1}}\right] + A_{\omega=\omega_{c1}}$

$= -20 \times \log\frac{0.5}{0.25} + 14 = +8$ dB.

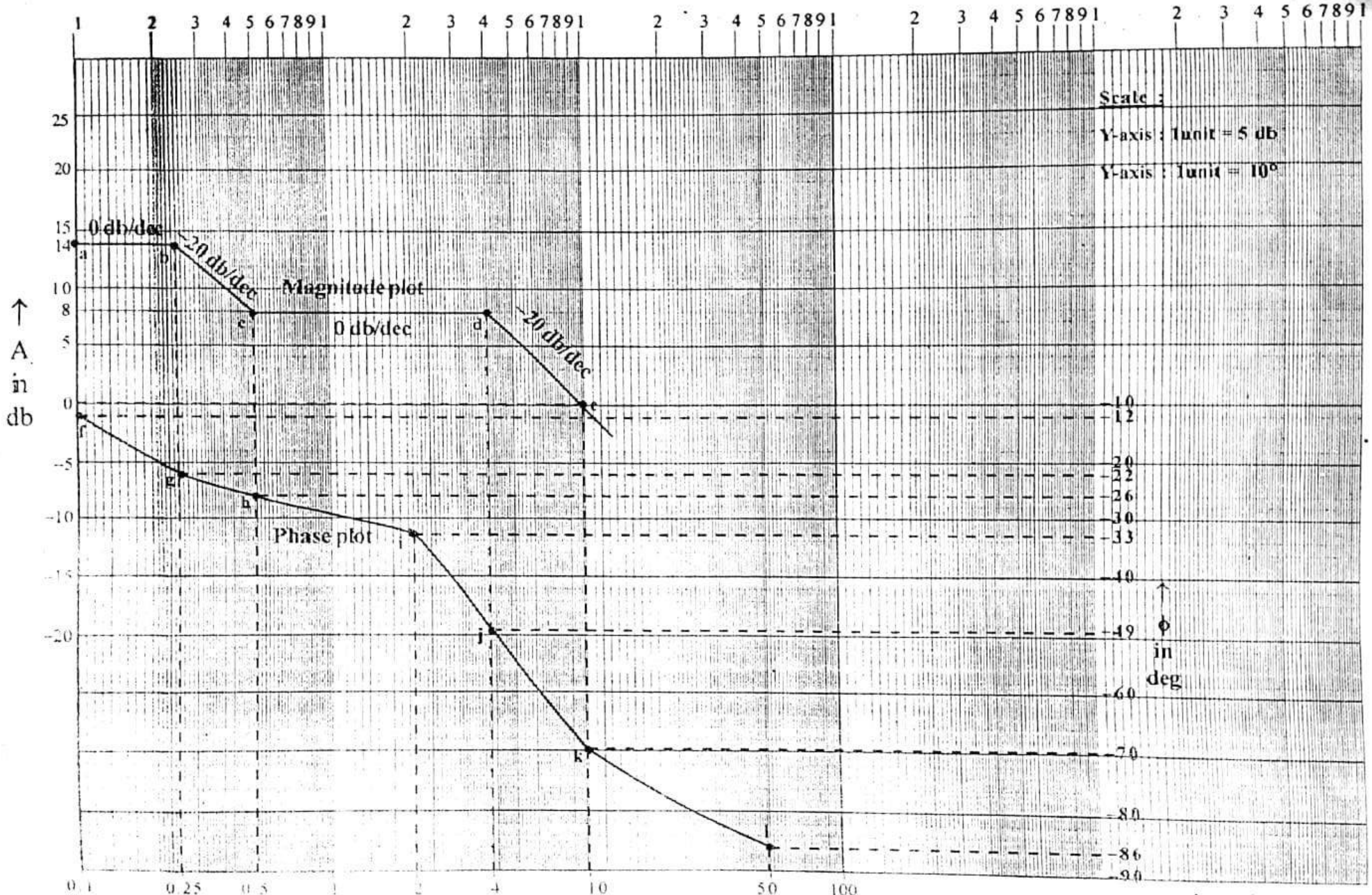


Fig 4.6.1.: Bode plot of transfer function, $G(j\omega) = \dfrac{5\,(1 + j2\omega)}{(1 + j4\omega)\,(1 + j0.25\omega)}$

At $\omega = \omega_{c3}$, $A = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log \frac{\omega_{c3}}{\omega_{c2}}\right] + A_{(\omega = \omega_{c2})}$

$$= 0 \times \log \frac{4}{0.5} + 8 = +8\,dB.$$

At $\omega = \omega_h$, $A = \left[\text{slope from } \omega_{c3} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c3}}\right] + A_{\omega = \omega_{c3}}$

$$= -20 \log \frac{10}{4} + 8 = 0\,dB.$$

Sno	ω	ω-Values	magnitude	points
1	ω_l	0.1	+14	a
2	ω_{c1}	0.25	+14	b
3	ω_{c2}	0.5	+8	c
4	ω_{c3}	4	+8	d
5	ω_h	10	0	e

Phase plot:

$$\phi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$$

ω	$\tan^{-1} 2\omega$	$\tan^{-1} 4\omega$	$\tan^{-1} 0.25\omega$	ϕ	Points
0.1	11.3	21.8	1.43	-12	f
0.25	26.56	45.0	3.5	-22	g
0.5	45.0	63.43	7.1	-26	h
2	75.96	82.87	26.56	-33	i
4	82.87	86.42	45.0	-49	j
10	87.13	88.56	68.19	-70	k
50	89.42	89.71	85.42	-86	l.

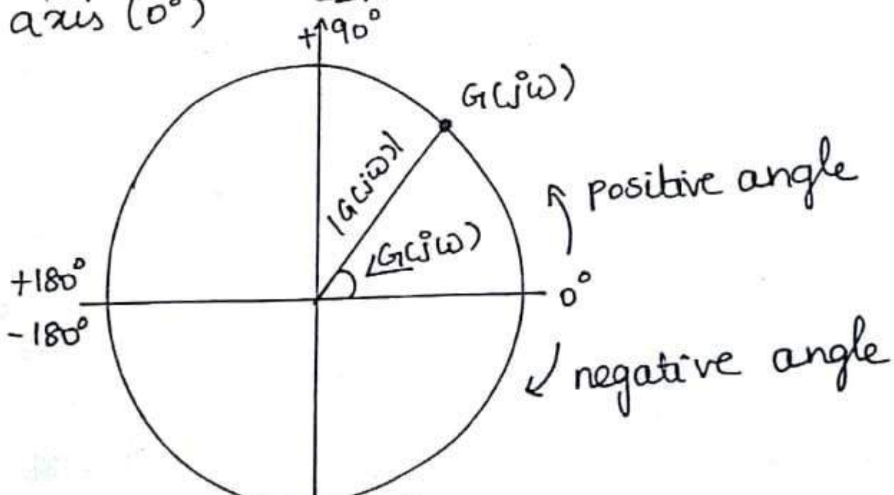
POLAR PLOT

The polar plot of sinusoidal transfer function $G(j\omega)$ is a plot of magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. Thus polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

Polar plot is the base of Nyquist plot and the stability analysis using Nyquist plot method. It is not necessary to convert magnitude to its dB value or find logarithm of frequencies.

Polar plot is usually plotted on polar graph sheet. This sheet has concentric circles and radial lines. concentric circles represent magnitude and radial lines represent phase angles.

In that sheet, positive phase angle is measured in anticlockwise from reference axis (0°) and negative angle is measured clockwise from reference axis (0°)

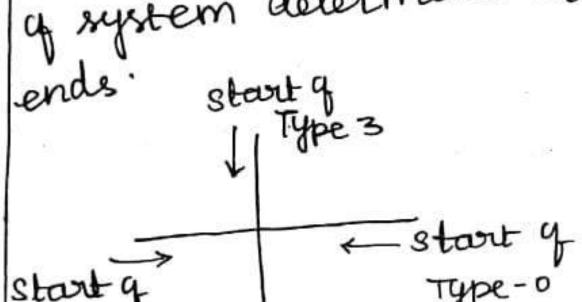


Alternative if $G(j\omega)$ can be expressed in rectangular coordinates,

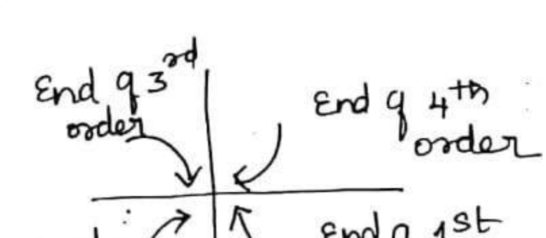
$$G(j\omega) = \underbrace{G_R(j\omega)}_{\text{Real part of } G(j\omega)} + j\,\underbrace{G_I(j\omega)}_{\text{Img. part of } G(j\omega)}$$

Then, polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞. In order to plot the polar plot on ordinary graph sheet, magnitude & phase of $G(j\omega)$ are computed for various values of ω. Then convert polar coordinates to rectangular coordinates using $P \to R$ in calculator.

For minimum phase transfer function with only poles, type number of the system determines the quadrant at which polar plot starts and order of system determines quadrant at which polar plot ends.



Start of polar plot



End of polar plot

The change in shape of polar plot can be predicted due to addition of pole or zero.

(i) when a pole is added to a system, polar plot end point will shift by $-90°$

(ii) when a zero is added to a system, polar plot end point will shift by $+90°$

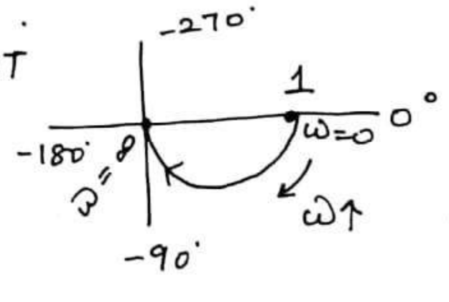
Typical sketches of polar plot

(i) $\boxed{G(s) = \frac{1}{1+sT}}$

$\Rightarrow$ Type: 0, order: 1

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T}$$

$$G(j\omega) = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T$$



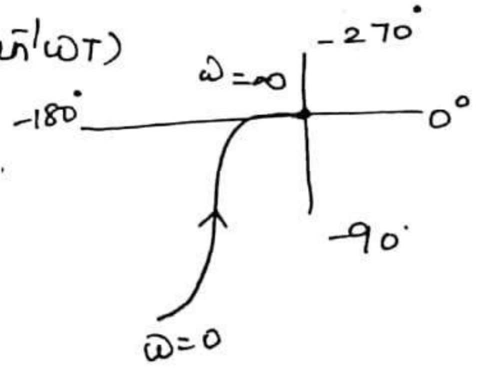
$\omega \to 0, \; G(j\omega) = 1\angle 0°$

$\omega \to \infty, \; G(j\omega) = 0\angle -90°$

(ii) $\boxed{G(s) = \frac{1}{s(1+sT)}} = \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega\angle 90° \sqrt{1+\omega^2T^2}\angle \tan^{-1}\omega T}$

$\Rightarrow$ Type: 1, order: 2

$$G(j\omega) = \frac{1}{\omega\sqrt{1+\omega^2T^2}} \angle(-90° - \tan^{-1}\omega T)$$



$\omega = 0, \; G(j\omega) = \infty \angle -90°$

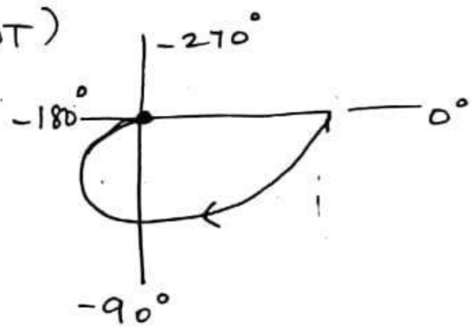
$\omega = \infty, \; G(j\omega) = 0\angle -180°$

(iii) $\boxed{G(s) = \frac{1}{(1+sT_1)(1+sT_2)}}$

Type: 0, order: 2

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2 T_1^2}\,\angle \tan^{-1}\omega T_1 \,\sqrt{1+\omega^2 T_2^2}\,\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2 T^2}} \angle(-90^\circ - \tan^{-1}\omega T)$$



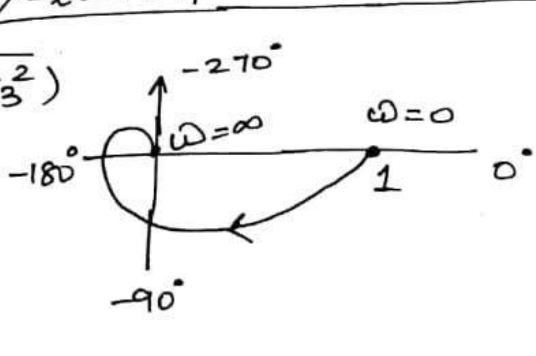
As $\omega \to 0$ $G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty$. $G(j\omega) = 0\angle -180^\circ$

(iv) $\boxed{G(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}}$

Type: 0, order: 3

$$G(j\omega) = \frac{\angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3}{\sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}}$$



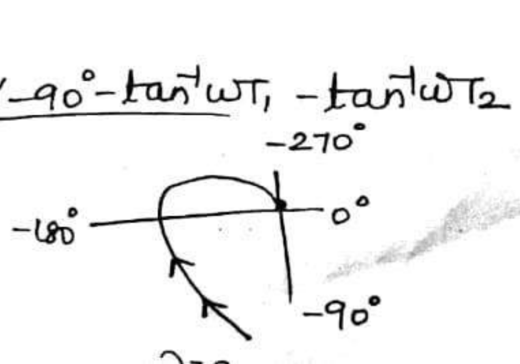
As $\omega \to 0$, $G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty$, $G(j\omega) = 0\angle -270^\circ$

(v) $\boxed{G(s) = \frac{1}{s(1+sT_1)(1+sT_2)}}$

Type: 1; order: 3

$$G(j\omega) = \frac{1}{\omega\sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$



$\omega = 0$, $G(j\omega) = \infty \angle -90^\circ$

$\omega = \infty$, $G(j\omega) = 0 \angle -270^\circ$

(vi) $\boxed{G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}}$

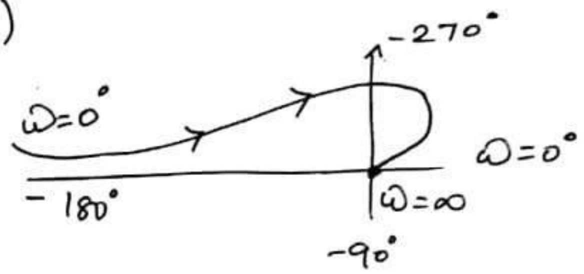
Type: 2, order: 4

$$G(j\omega) = \frac{1}{\omega^2 \angle{-180^\circ} \sqrt{1+\omega^2 T_1^2} \angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2 T_2^2} \angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle(-180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

$\omega = 0,\ G(j\omega) = \infty \angle{-180^\circ}$

$\omega = \infty,\ G(j\omega) = 0 \angle{-360^\circ}$



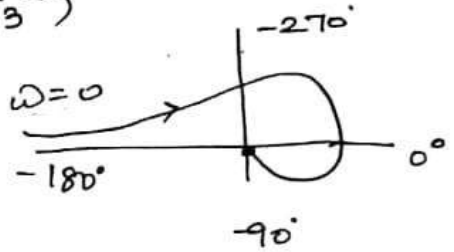
(vii) $\boxed{G(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}}$

Type: 2, order: 5

$$G(j\omega) = \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}} \angle{-180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3}$$

$\omega = 0,\ G(j\omega) = \infty \angle{-180^\circ}$

$\omega = \infty,\ G(j\omega) = 0 \angle{-450^\circ} = 0 \angle{-90^\circ}$

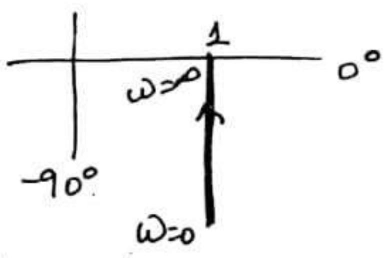


(viii) $\boxed{G(s) = \frac{1+sT}{sT}}$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T} + 1 = \frac{1}{\omega T \angle 90^\circ} + 1 = \frac{1}{\omega T} \angle{-90^\circ} + 1$$

$\omega = 0,\ G(j\omega) = \infty \angle{-90^\circ} + 1$

$\omega = \infty,\ G(j\omega) = 0 \angle{-90^\circ} + 1$



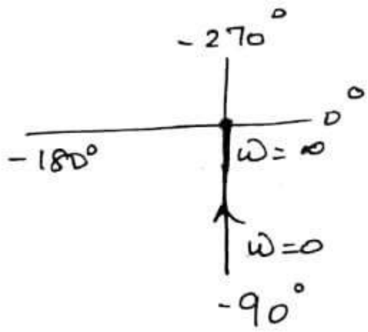
(ix) $\boxed{G(s) = 1/s}$

$\Rightarrow$ Type: 1, order: 1

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega}\angle -90^\circ$$

At $\omega = 0$, $G(j\omega) = \infty \angle -90^\circ$

At $\omega = +\infty$, $G(j\infty) = 0 \angle -90^\circ$



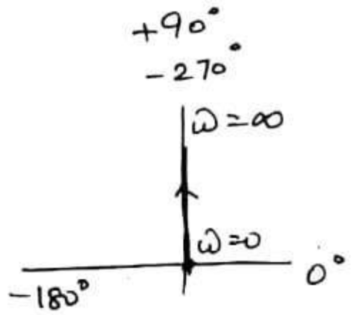
x) $\boxed{G(s) = s}$

$= \quad G(s) = s$

$= G(j\omega) = j\omega = \omega\angle 90^\circ$

At $\omega = 0$, $G(j\omega) = 0\angle 90^\circ$

At $\omega = \infty$, $G(j\omega) = \infty \angle 90^\circ$



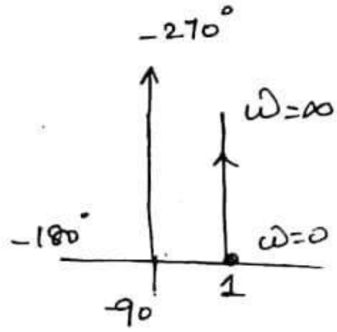
xi) $\boxed{G(s) = 1 + sT}$

$G(j\omega) = 1 + j\omega T$

$G(j\omega) = 1 + \omega T \angle 90^\circ$

At $\omega = 0$, $G(j\omega) = 1 \angle 90^\circ$

$\omega = \infty$, $G(j\omega) = 1 + \infty \angle 90^\circ$



— X —

NO/DEC 2014. $G(s) = \frac{1}{s(s+4)(s+8)}$ 21

Qb 2

1. The open loop transfer function of a unity feedback system is given by $G(s) = 1/s(1+s)(1+2s)$ sketch polar plot and determine gain margin and phase margin.

polar to rectangular
Real = mag. cos(phase)
imag = mag. sin(phase)

= solution :- $$G(s) = \frac{1}{s(1+s)(1+2s)}$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)}$$

corner frequencies = $\omega_{c1} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c2} = 1$ rad/sec.

$\omega_l < \omega_{c1}$ & $\omega_h > \omega_{c2}$.

Phase angle :-

$$G(j\omega) = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} \quad \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

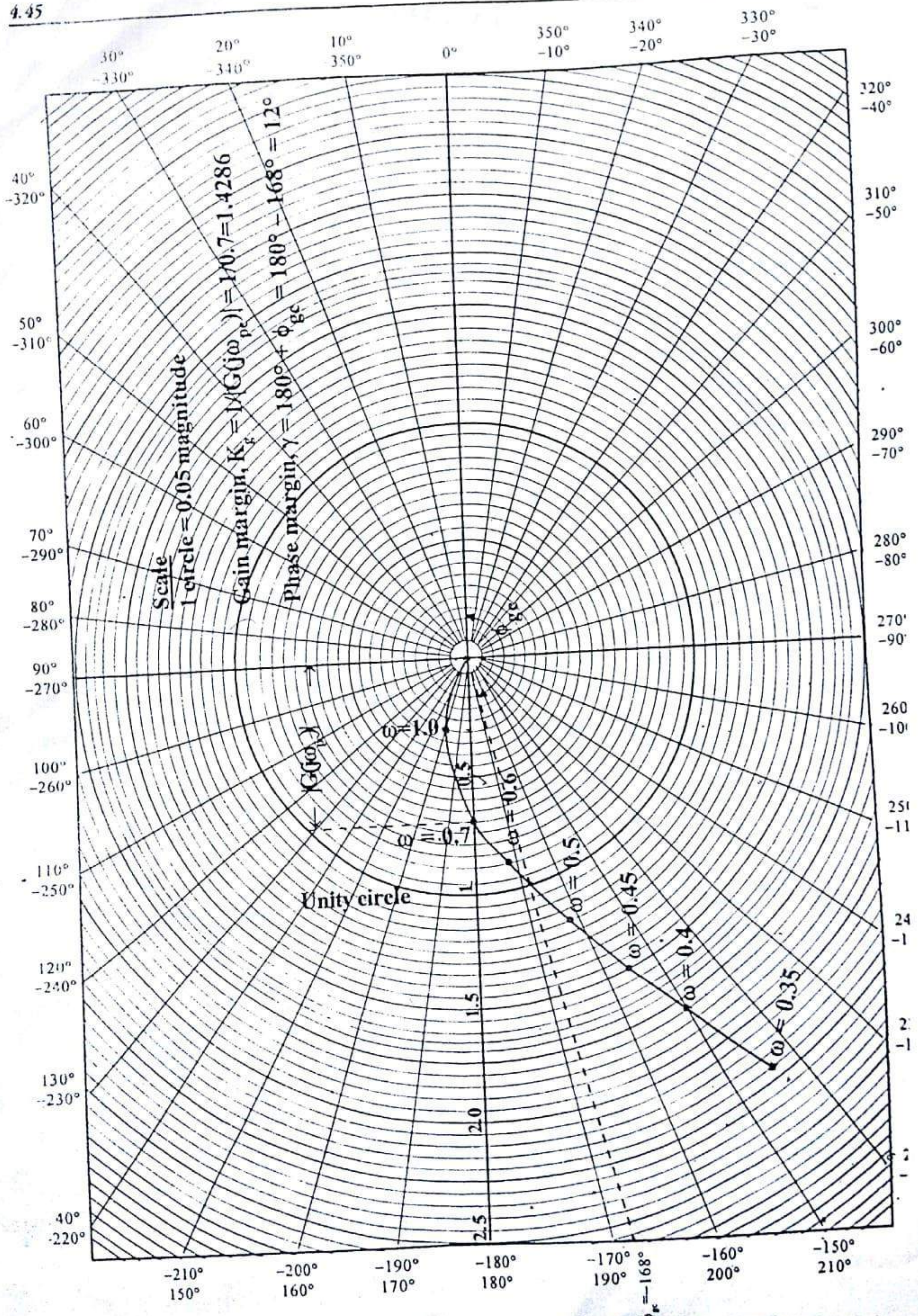
Magnitude & phase of $G(j\omega)$ at various freq.

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$	-144	-150	-156	-162	-171	-180°	-198

Real & Imaginary part of $G(j\omega)$ at Various freq

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

Result: Gain Margin : $k_g = 1.4286$; Phase margin $\gamma = +12^\circ$.


Scale
1 circle = 0.05 magnitude
Gain margin, $K_g = 1/|G(j\omega_{pc})| = 1/0.7 = 1.4286$
Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 168° = 12°$
Unity circle
$|G(j\omega_{pc})|$
$\omega = 1.0$
$\omega = 0.7$
$\omega = 0.6$
$\omega = 0.5$
$\omega = 0.45$
$\omega = 0.4$
$\omega = 0.35$
ϕ_{gc}
$\phi_g = -168°$
0.5
1
1.5
2.0
2.5

NYQUIST PLOT

The concept of Nyquist plot is based on the polar plot which can be conveniently applied to the stability analysis of any kind of system.

[Nyquist stability criterion states that "If the Nyquist plot of open loop transfer function G(s) H(s) corresponding to Nyquist contour in the s-plane encircles the critical point $-1+j0$ in the counter clock-wise direction as many times as the number of right half s-plane poles of G(s) H(s), the closed loop system is stable".]

(Nyquist has suggested that rather than analyzing whether all zeros are located in left half of s-plane, it is better to examine the presence of any one zero of $1+G(s)H(s)$ in right half of s-plane making system unstable) Hence the active region from stability point of view is right half of s-plane. So instead of choosing any arbitrary path $\tau(s)$ in s plane, Nyquist has suggested to select a $\tau(s)$ path which will encircle the entire right half of s-plane.

Such a path should start from $s=+j\infty$. It should be continued till $s=-j\infty$ along imaginary axis and should be completed with a semicircle of radius ∞, encircling entire right half of s-plane. This path is called Nyquist path or Nyquist Contour.

Steps to solve problems by Nyquist criteria

Step 1 :-

count how many number of poles of $G(s)H(s)$ are in the right half of s-plane. ie., positive real part. This is the value of P.

Step 2 :-

Decide the stability criterion as $N = -P$ ie., how many times Nyquist plot should encircle $-1+j0$ point for absolute stability.

Step 3 :-

select Nyquist path as per function $G(s)H(s)$.

Step 4 :-

Analyse the sections as starting point and terminating point of plot.

Step 5 :-

Mathematically find out ω_{pc} and intersection of Nyquist plot with negative real axis by rationalizing $G(j\omega)H(j\omega)$

Step 6 :-

with the knowledge of step 4 & 5, sketch Nyquist plot

Step 7 :-

count the number of encirclements N of $(-1+j0)$ by Nyquist plot. If this matches with criterion decided in step 2, system is stable otherwise unstable.

— x — x —

1. A unity feedback control system has

$$G(S) = \frac{10}{S(S+1)(S+2)}$$. Draw Nyquist plot and comment on closed loop stability.

= <u>Given</u>: $G(S)H(S) = \frac{10}{S(S+1)(S+2)}$ $(\because H(S)=1)$

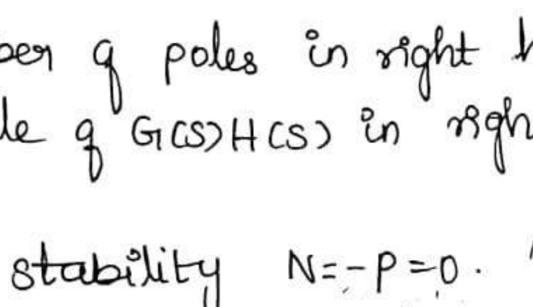
<u>step 1</u>:-

0, -1, -2

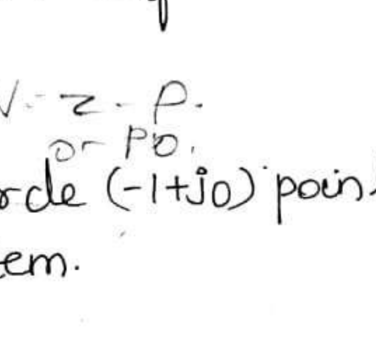
Number of poles in right half $P=0$.
No pole of $G(S)H(S)$ in right half.

<u>step 2</u>:-
For stability $N = -P = 0$. $N = Z - P$. or P0.
ie., Nyquist plot should not encircle $(-1+j0)$ point for absolute stability of this system.

<u>step 3</u>:
As there is one pole at origin, it should be bypassed by semicircle (small).



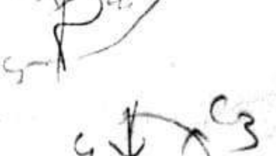
Ny criteria
N, K



Nyquist Contours

<u>step 4</u>:- $G(j\omega)H(j\omega) = \frac{10}{j\omega(1+j\omega)(2+j\omega)}$

$$|G(j\omega)H(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$$

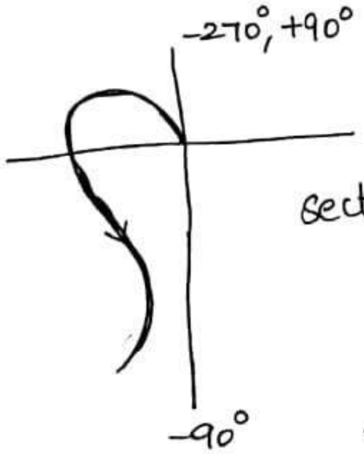


$$\phi = \frac{\cancel{\tan^{-1}\left(\frac{0}{10}\right)}^{\,0}}{\tan^{-1}\left(\frac{\omega}{0}\right)\tan^{-1}\left(\frac{\omega}{1}\right)\tan^{-1}\left(\frac{\omega}{2}\right)} = -90^\circ - \tan^{-1}\omega - \tan^{-1}\omega/2$$

<u>Section : I</u> $\boxed{S = +j\infty}$ to $\boxed{S = +j0}$ ie., $\omega \to \infty$ to $\omega \to 0$

starting point : $\omega \to \infty \Rightarrow 0\angle{-270^\circ}$ — $-90^\circ - (-270^\circ) = +180^\circ$ Anticlock wise rotation.

terminating point : $\omega \to 0 \Rightarrow \infty\angle{-90^\circ}$

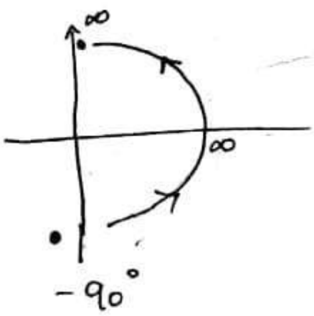


section - I mapping.

<u>Section : II</u> $\boxed{S = +j0}$ to $\boxed{S = -j0}$ ie., $\omega \to +0$ to $\omega \to -0$

Starting point : $\omega \to +0 \Rightarrow \infty\angle{-90^\circ}$ — $90^\circ - (-90^\circ) = +180^\circ$ Anticlock wise Rotation.

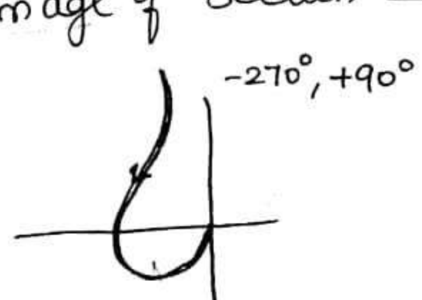
Terminating : $\omega \to -0 \Rightarrow \infty\angle{+90^\circ}$



section II mapping

<u>Section : III</u> is the mirror image of Section - I about real axis

section - III mapping



Section : IV It is an origin and not required to analysed.

Step 5 :-

Find out intersection with negative real axis.

Rationalise, $G(j\omega)H(j\omega)$.

$$G(j\omega)H(j\omega) = \frac{10(-j\omega)(1-j\omega)(2-j\omega)}{(j\omega)(-j\omega)(1+j\omega)(1-j\omega)(2+j\omega)(2-j\omega)}$$

$$= \frac{-30\omega^2 - 10j\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

separate real & Img part.

$$G(j\omega)H(j\omega) = \frac{-30\omega^2}{\omega^2(1+\omega^2)(4+\omega^2)} - j\frac{10\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

At $\omega = \omega_{pc} \Rightarrow$ Imaginary part is zero.

$$\therefore 10\omega(2-\omega^2) = 0 \Rightarrow \omega = 0, \sqrt{2}.$$

$\therefore \boxed{\omega_{pc} = \sqrt{2}}$ as it has to be positive

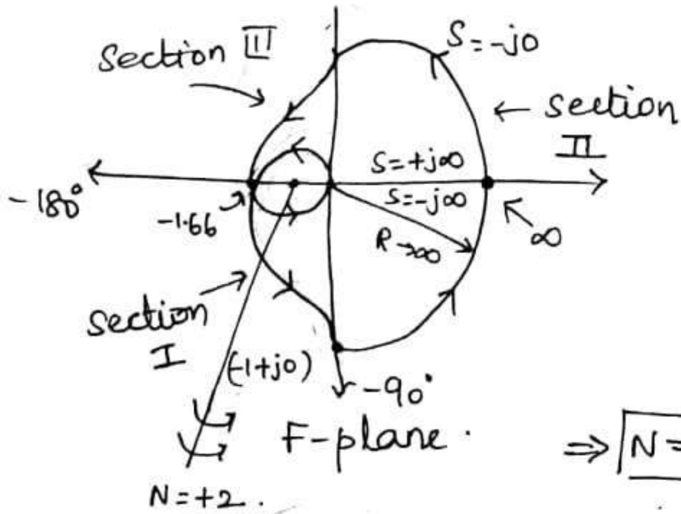
Substitute $\omega_{pc} = \sqrt{2}$ in real part.

$$G(j\omega)H(j\omega)\Big|_{\omega=\omega_{pc}} = \frac{-30\times 2}{2(1+2)(4+2)} + j0 = -1.66 + j0.$$ $-1.66 - j0$

Gain margin : $\frac{1}{|-1.66+j0|} = \frac{1}{1.66} = 0.6 = \boxed{-4.43\,dB}$.

As GM is negative, System is unstable because critical point is enclosed.

Step 6:- Nyquist plot



$\Rightarrow \boxed{N=+2}$

Step 7:-

The number of encirclement of $-1+j0$ are $\boxed{N=+2}$, but as per Step 2: $\boxed{N=0}$ Hence it does not match. ∴ given system is unstable.

According to Mapping theorem, $\boxed{N=Z-P}$

$2=Z-0 \quad \therefore \boxed{Z=2}$

ie., Actually there are 2 zeros of $1+G(s)H(s)$ encircled by Nyquist path. ie., 2 closed loop poles in right half of S-plane due to which closed loop system is unstable.

Note:- For all system with $G(s)H(s) = \dfrac{K}{S(1+T_1S)(1+T_2S)}$, the shape of Nyquist plot will remain same as shown in above diagram. only coordinates of point A will change depending on values of K, T_1, T_2 which will decide encirclements of $-1+j0$ and hence the stability.

—x —x—

Advantage of Nyquist plot :-

1. It gives same information about absolute stability as provided by Routh's criteria.

2. useful for determining the stability of closed loop system from OLTF without knowing roots of C.E.

3. It also indicates relative stability giving the values of G.M and P.M.

4. It indicates reality, the manner in which system should be compensated to yield desired response.

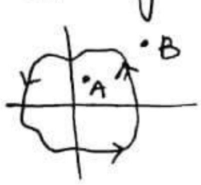
5. Information regarding frequency response can be obtained.

6. Very useful for analyzing conditionally stable systems.

— x —

Encircled :-

A point or region in a complex function plane is said to be encircled by a closed path if it is found inside the path.



Enclosed :-

A point or region is said to be enclosed by closed path if it is encircled in the counter clockwise, or point or region lies to left of the path when the path is traversed in prescribed direction

A enclosed by Γ ⇒ 

⇐ A is not enclosed by Γ

DESIGN AND COMPENSATION

It has often been observed that the performance of a control system does not satisfy the given specifications in terms of accuracy, stability, damping, speed response and so on. After design and testing if the system does not perform satisfactorily some changes may need to be introduced to achieve the desired results. The changes could be in form of adjustment of of forward path gain or insertion of compensating device in the control system.

To reduce the steady state error, gain can be increased. However, it results in an oscillatory transient response or even instability. under such circumstances, it may be necessary to introduce some kind of corrective subsystems to force the chosen plant to meet the given specifications. These subsystems are known as compensators. and their job is to compensate for deficiency in the performance of plant.

compensators are required in following two cases. namely,

(i) System is unstable, compensation is required to stabilize it and also achieve the desired performance specifications.

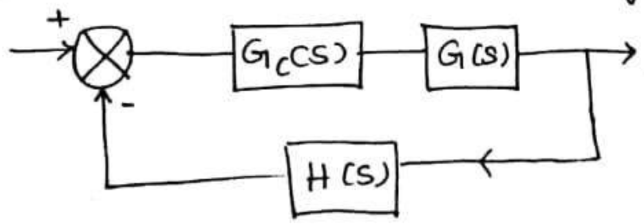
(ii) system is ~~stable~~, compensation is required to achieve the improved performance specifications.

Compensation by Inserting N/W

- Series compensation (or) Cascade compensation
  - cascade lag N/W (Phase lag occurs in f_L)
  - cascade lead N/W (Phase leads occurs in f_H)
  - cascade lag-lead. N/W
- parallel compensation (or) feedback compensation
- series-parallel compensation.

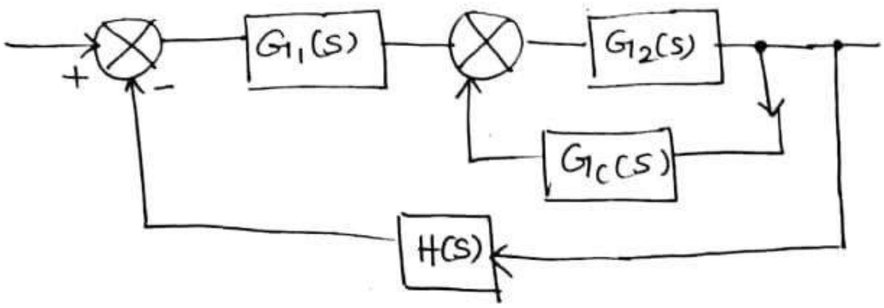
Cascade compensation :-

If the compensator $G_c(s)$ is placed in series with forward path transfer function of the plant, the scheme is called series or cascade compensation. The flow of signal in such series is from lower energy level towards higher energy level. This requires additional amplifiers to increase the gain and also to provide necessary isolation.

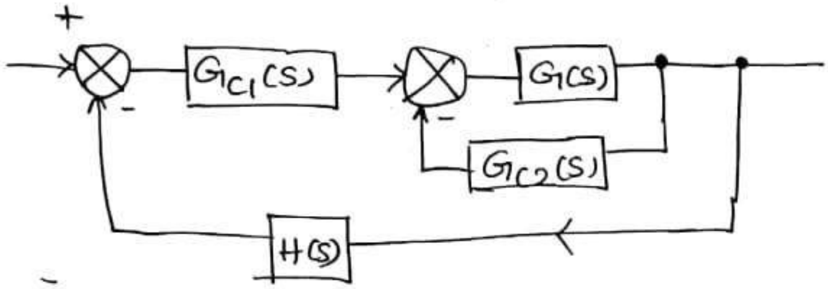


Feedback compensation :-

If compensator $G_c(s)$ is placed in feedback path to provide an additional internal feedback loop. The energy transfer is from higher energy level towards lower energy level point. Additional Amplifiers are not required.



Series-parallel compensation:-

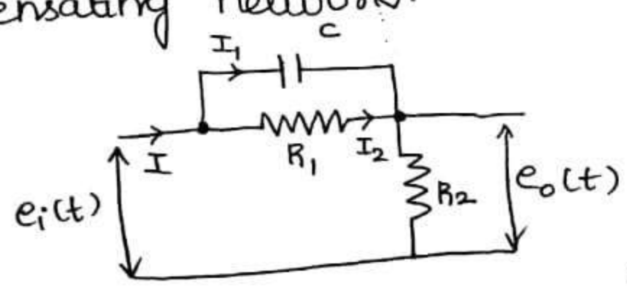


The selection of proper compensation scheme depends on nature of signals available in the system, power levels at various points, available components, economic considerations, and designer's Experience.

—x——x—

LEAD COMPENSATOR

Consider an electrical RC Network which is lead compensating network.



Transfer function : $G_C(S) = \dfrac{E_o(S)}{E_i(S)} =$

Apply KCL, $\boxed{I_1 + I_2 = I}$

$$\frac{c\,d(e_i - e_o)}{dt} + \frac{1}{R_1}(e_i - e_o) = \frac{1}{R_2} e_o(t)$$

Take laplace transform on both sides

$$Sc\left[E_i(S) - E_o(S)\right] + \frac{1}{R_1}\left[E_i(S) - E_o(S)\right] = \frac{1}{R_2} E_o(S)$$

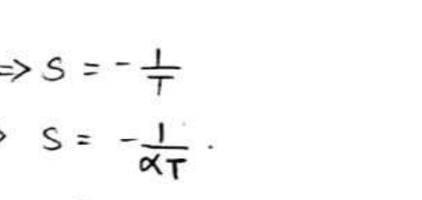
$$E_i(S)\left[SC + \frac{1}{R_1}\right] = E_o(S)\left[SC + \frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$T.F = \frac{E_o(S)}{E_i(S)} = \frac{R_1 R_2}{R_1 + R_2 + R_1 R_2 SC}\left(\frac{1 + SCR_1}{R_1}\right)$$

$$\frac{E_o(S)}{E_i(S)} = \frac{\left(S + \frac{1}{R_1 C}\right)}{S + \frac{(R_1 + R_2)}{R_1 R_2 C}} = \frac{\left(S + \frac{1}{R_1 C}\right)}{\left[S + \frac{1}{\left(\frac{R_2}{R_1 + R_2}\right) R_1 C}\right]}$$

This is generally expressed as,

$$\boxed{\frac{E_o(S)}{E_i(S)} = \frac{S + \frac{1}{T}}{S + \frac{1}{\alpha T}}}$$



where $T = R_1 C$, $\alpha = R_2/R_1 + R_2 < 1$

In lead compensator has (i) zero $\Rightarrow S = -\frac{1}{T}$
(ii) pole $\Rightarrow S = -\frac{1}{\alpha T}$.

Maximum lead angle ϕ_m and α :-

$T(ST+1)$

$$\frac{E_o(S)}{E_i(S)} = \frac{S + 1/T}{S + 1/\alpha T} = \frac{\alpha(1 + TS)}{(1 + \alpha TS)}$$

Replace S by $j\omega \Rightarrow \frac{E_o(j\omega)}{E_i(j\omega)} = \frac{\alpha(1 + j\omega T)}{(1 + \alpha j\omega T)}$

$$\left|\frac{E_o(j\omega)}{E_i(j\omega)}\right| = M = \frac{\alpha\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2\alpha^2T^2}}$$

while phase angle is given by,

$$\phi = \tan^{-1}\omega T - \tan^{-1}\omega\alpha T.$$

then $\frac{d\phi}{d\omega} = 0 \Rightarrow$ gives ω_m (maximum frequency)

$$\frac{d}{d\omega}\left[\tan^{-1}\omega T - \tan^{-1}\omega\alpha T\right] = 0$$

$$= \frac{\frac{1}{T}}{\omega^2+\left(\frac{1}{T}\right)^2} - \frac{(1/\alpha T)}{\omega^2+\left(1/\alpha T\right)^2} = 0$$

$$= \frac{T}{1+\omega^2T^2} - \frac{\alpha T}{1+\alpha^2\omega^2T^2} = 0$$

$$T(1+\alpha^2\omega^2T^2) - \alpha T(1+\omega^2T^2) = 0.$$

$$1+\alpha^2\omega^2T^2 - \alpha - \alpha\omega^2T^2 = 0.$$

$$\omega^2\alpha T^2(\alpha-1) + (1-\alpha) = 0.$$

$$\omega^2 = \frac{1}{\alpha T^2} \Rightarrow \boxed{\omega_m = \frac{1}{T\sqrt{\alpha}} = \sqrt{\left(\frac{1}{T}\right)\left(\frac{1}{\alpha T}\right)}}$$

This is the frequency at which phase lead is at its maximum. The corner frequencies of compensator,

$$\boxed{\omega_{c1} = \frac{1}{T}\ ;\ \omega_{c2} = \frac{1}{\alpha T}}$$

Taking tan on both sides of phase angle,

$$\tan\phi = \tan\left[\tan^{-1}\omega T - \tan^{-1}\alpha\omega T\right]$$

$$= \frac{\omega T - \alpha\omega T}{1+\omega T\cdot\alpha\omega T} = \frac{\omega T(1-\alpha)}{1+\omega^2T^2\alpha}$$

$$\tan\phi = \frac{\omega T(1-\alpha)}{1+\omega^2T^2\alpha}$$

At $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$

$$\tan\phi = \frac{\omega T(1-\alpha)}{1+\omega^2T^2\alpha}$$

$$\tan\phi_m = \frac{\omega_m T(1-\alpha)}{1+\omega_m^2T^2\alpha} \Rightarrow \frac{1-\alpha}{\sqrt{\alpha}(1+1)} = \frac{1-\alpha}{2\sqrt{\alpha}}.$$

$$\boxed{\sin\phi_m = \frac{1-\alpha}{1+\alpha}}$$

This equation is also used to get the relation between α and maximum lead angle ϕ_m.

Bode plot of lead compensator.

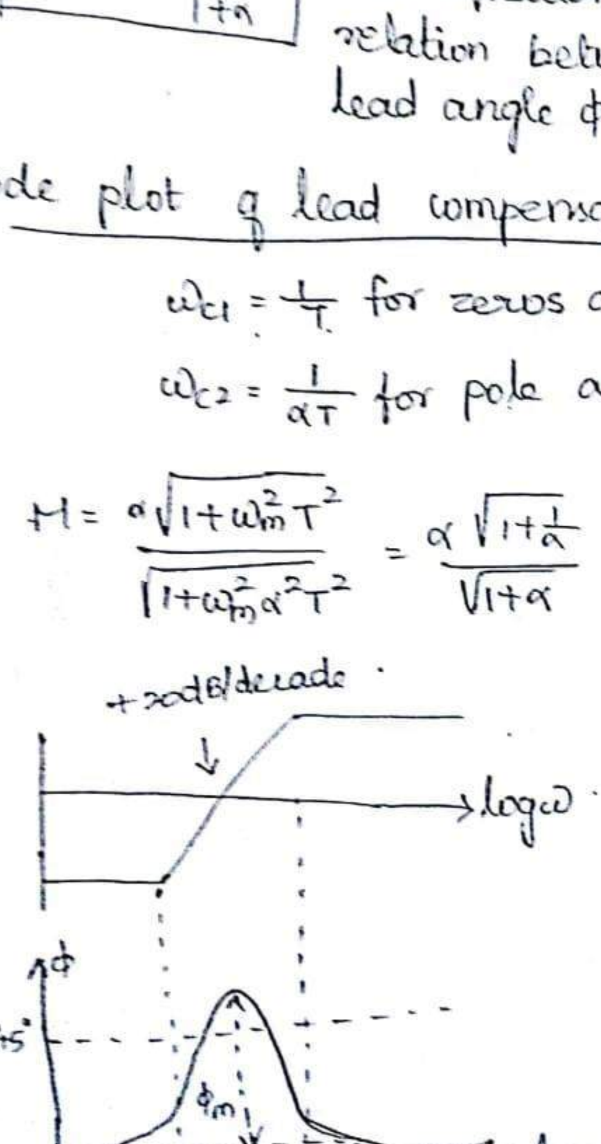
$\omega_{c1} = \frac{1}{T}$ for zeros at $s = -\frac{1}{T}$

$\omega_{c2} = \frac{1}{\alpha T}$ for pole at $s = -\frac{1}{\alpha T}$. & $K = \alpha$.

$$M = \frac{\alpha\sqrt{1+\omega_m^2T^2}}{\sqrt{1+\omega_m^2\alpha^2T^2}} = \frac{\alpha\sqrt{1+\frac{1}{\alpha}}}{\sqrt{1+\alpha}} \Rightarrow \boxed{M = \sqrt{\alpha}} \text{ at } \omega = \omega_m.$$

$$= 20\log(\alpha)^{1/2} = 10\log\alpha$$

$$M = -10\log\left(\frac{1}{\alpha}\right) dB.$$



Steps to design lead compensator

step 1:-

At zero frequency, lead compensator has gain α. But as $\alpha < 1$, it provides attenuation.

$$G_c(s) = \cancel{K_c \alpha} \frac{(1+Ts)}{(1+\alpha Ts)} = \frac{K(1+Ts)}{(1+\alpha Ts)}$$

open loop transfer function of compensated system,

$$G_c(s)\,G(s) = \frac{K(1+Ts)}{(1+\alpha Ts)} \cdot G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} \cdot K\,G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} G_1(s)$$

where $\boxed{G_1(s) = KG(s)}$

Determine the value of K satisfying given error constant.

step 2:-

Draw the Bode plot $G_1(j\omega)$ with K. gives uncompensated system. obtain the phase margin. 'ϕ_1'

step 3:-

let ϕ_s = PM specified

ϕ_1 = PM obtained in step 2.

$\therefore \boxed{\phi_m = \phi_s - \phi_1 + \varepsilon}$ where $\varepsilon_p = 5^\circ$ to 15°
= Margin of safety as cross over frequency.

step 4:- $\boxed{\sin\phi_m = 1-\alpha/1+\alpha}$.

determine value of α.

Step 5 :-

Determine frequency ω_m at which magnitude of uncompensated system is $-10\log\left(\frac{1}{\alpha}\right)$ dB. select this frequency as new gain crossover frequency.

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$ as 'α' is known, determine $\frac{1}{T}$

Step 6 :-

Determine two frequencies of lead compensator.

$$\omega_{c1} = \frac{1}{T} \quad \& \quad \omega_{c2} = \frac{1}{\alpha T}.$$

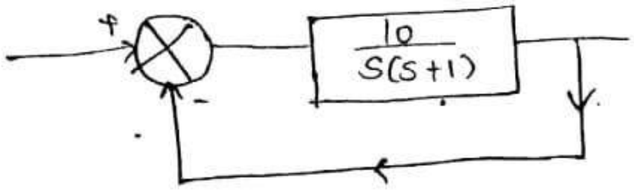
Step 7 :- As $K = K_c\alpha$. determine value of K_c.

Step 8 :- check the gain margin of compensated system. If it is not satisfactory, repeat the design by modifying pole-zero location of compensator till a satisfactory result is obtained.

—— x —— x ——

Problems on Lead compensator

1. For the system shown in Figure, design a lead compensator such that closed loop system will satisfy the following specifications. Static Velocity error constant = 80 sec^{-1}, phase margin = 50°, Gain margin ≥ 10dB.



Solution:- Step 1 :- Assume a lead compensator as,

$$G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)} = K \frac{(1+Ts)}{1+\alpha Ts}$$

$$G_1(s) = K\,G(s) = \frac{10K}{s(s+1)}$$

$$K_v = 20 \Rightarrow \lim_{s\to 0} s\,G(s)H(s) = \lim_{s\to 0} s\left[\frac{10K(1+Ts)}{s(s+1)(1+\alpha Ts)}\right]$$

$$20 = K10 \Rightarrow \boxed{K=2} \Rightarrow G_1(s) = \frac{20}{s(s+1)}$$

Step 2:- Sketch Bode plot $G_1(s)$.

factors: $20\log 20 = 26\,dB$

1 pole at origin

1 simple pole with corner frequency $\omega_c = 1$.

Thus line of slope $-20\,dB/dec$ till $\omega_c = 1$ and line of slope $-40\,dB/dec$ from 1 onwards.

phase angle table: $G_1(j\omega) = 20/j\omega(1+j\omega)$

ω	$1/j\omega$	$-\tan^{-1}\omega$	ϕ_R
0.1	$-90°$	$-5.71°$	-95.71
1	$-90°$	$-45°$	$-135°$
2	$-90°$	$-63.4°$	$-153.4°$
10	$-90°$	$-84.2°$	-174.2
α	$-90°$	$-90°$	$-180°$

from diagram,

$\phi_1 = PM = 15°$, $\omega_{gc} = 4\,rad/sec$

$GM = +\infty\,dB$.

Step 3:- $\phi_s = 50^\circ$ (given)

$\phi_m = \phi_s - \phi_1 - \varepsilon_\phi$ let $\varepsilon_\phi = 5^\circ$

$\phi_m = 50 - 15 - 5 = 40^\circ$

step 4:- $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$

$\sin 40^\circ = \frac{1-\alpha}{1+\alpha} = 0.6427$

$\Rightarrow \boxed{\alpha = 0.2174}$

Step 5: $\alpha = -10\log\left(\frac{1}{\alpha}\right) = -6.78\,dB$.

find the frequency at which gain of uncompen-sated system is $-6.78\,dB$ (from figure)

$\therefore \omega_m = 6$ rad/sec at gain $= -6.78\,dB$.

$\omega_m = \frac{1}{T\sqrt{\alpha}}$ ie., $\frac{1}{T} = 2.7495$.

step 6:

$\omega_{c1} = \frac{1}{T} = 2.7495$; $\omega_{c2} = \frac{1}{\alpha T} = 13.09$.

step 7:-

$K = K_c\alpha$

$K_c = \frac{K}{\alpha} = \frac{2}{0.21} = 9.523$

Step 8:

$$G_c(s) = 9.523 \times 0.21\left(\frac{1+0.3637s}{1+0.0763s}\right) = \frac{2\,(1+0.3637s)}{(1+0.0763s)}$$

This is the designed lead compensator.

$$G_c(S)\,G(S) = \frac{20\,(1+0.3637S)}{S(1+S)(1+0.0763S)}$$

Draw the Bode plot for this transfer function and obtain values of GM and PM.

Phase angle table:

ω	$1/j\omega$	$-\tan^{-1}\omega$	$\tan^{-1}0.3637\omega$	$-\tan^{-1}0.0763\omega$	ϕ
0.1	-90°	-5.71°	$+2.08^\circ$	-0.43°	-94.6°
1	-90°	-45	$+20^\circ$	-4.36	-119.36
2	-90°	-63.4	$+36^\circ$	-8.67	-126.07
10	-90°	-84.2	$+74^\circ$	-37.3°	-137.05
100	-90°	-89.4	$+88^\circ$	-82.53	-173.9°

From magnitude plot $\boxed{K=20}$

$\therefore$ $20\log 20 = 26\,dB$.

One pole at origin, straight line of slope $-20\,dB/dec$.

$\omega_{c1}=1$, slope $=-40\,dB/dec$ due to simple pole.

$\omega_{c2}=\frac{1}{T}=2.75$, slope $=-20\,dB/dec$ due to simple zero

$\omega_{c3}=\frac{1}{\alpha T}=13.09$, slope $=-40\,dB/dec$ due to simple pole.

from fig, $\omega_{gc}=7\,rad/sec$, $PM=+50^\circ$, $GM=+\infty\,dB$.

Thus compensated systems satisfy all specification

—— x —— x ——

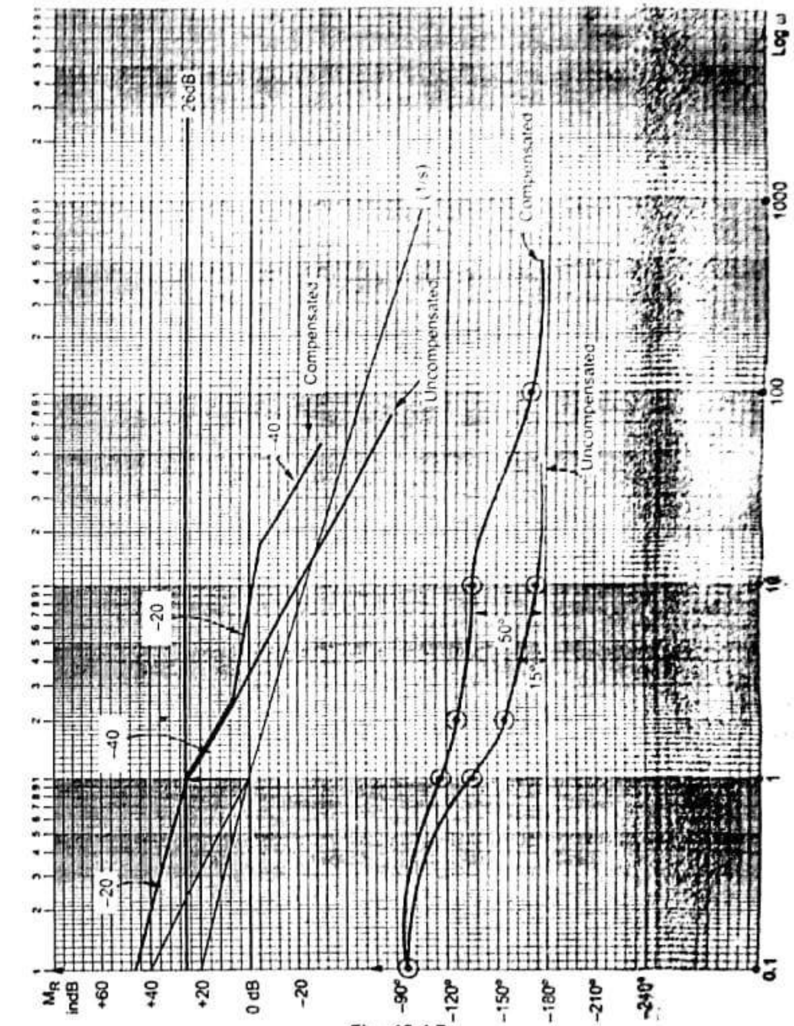


Fig. 13.4.7

Phase angle table for the compensated system :

[illegible]	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	$+\tan^{-1}0.3637\,\omega$	[illegible]	[illegible]
[illegible]	$-90°$	$-5.71°$	$+2.08°$	[illegible]	[illegible]

[illegible]	-90°	[illegible]	$+20^\circ$	[illegible]	-119.36°
[illegible]	-90°	[illegible]	$+36^\circ$	[illegible]	-126.07°
10	-90°	[illegible]	$+74^\circ$	[illegible]	-137.5°
100	-90°	-89.4°	$+88^\circ$	-82.53°	-173.9°

For magnitude plot, K = 20

$\therefore$ $20 \text{ Log } 20 = 26 \text{ dB}$

One pole at origin, straight line of slope – 20 dB/dec.

$\omega_{C1} = 1$, slope becomes – 40 dB/dec due to simple pole.

$\omega_{C2} = \frac{1}{T} = 2.75$, slope becomes – 20 dB/dec due to simple zero.

$\omega_{C3} = \frac{1}{\alpha T} = 13.09$, slope becomes – 40 dB/dec due to simple pole.

From the Fig. 13.4.7, for compensated system

$$\omega_{gc} = 7 \text{ rad/sec, P.M.} = +50^\circ, \text{ G.M.} = +\infty \text{ dB}$$

Thus the compensated system satisfies all the specifications.

Example 13.4.2 *Consider a type 1 unit feedback system with an OLTF $G_f = \frac{K}{s(s+1)}$. It is specified that $K_v = 12 \text{ sec}^{-1}$ and $\phi_{PM} = 40^\circ$. Design lead compensator to meet the specifications.* **AU : Dec.-04,05,06,08,09, May-04,05,12,13, Marks 16**

Solution :

Step 1 : Lead compensator $G_c(s) = K_c \alpha \frac{(1+Ts)}{(1+\alpha Ts)} = \frac{K(1+Ts)}{(1+\alpha Ts)}$

$$\therefore \quad G_c(s)G(s) = \frac{(1+Ts)}{(1+\alpha Ts)} \times \frac{K}{s(s+1)} \quad \text{where } G(s) = \frac{1}{s(s+1)}$$

$$\text{Now} \quad K_v = \lim_{s \to 0} sG_c(s)\,G(s) = \lim_{s \to 0} \frac{K}{s(s+1)} \frac{(1+Ts)}{(1+Ts)} = K$$

$$\therefore \quad K = 12$$

$$\therefore \quad G_1(s) = \frac{12}{s(s+1)}$$

Step 2 : Sketch the Bode plot of $G_1(s)$ i.e. uncompensated system as shown in the Fig. 13.4.8 (a). (See Fig. 13.4.8 (a) on page 13-16)

Factors : K = 12 i.e. 20 Log K = 21.58 ≈ 22 dB

1. Pole at the origin i.e. – 20 dB/dec

1 simple pole with $\omega_{C1} = 1$ i.e. – 20 dB/dec for $\omega > 1$.

So resultant starting slope – 20 dB/dec and then – 40 dB/dec for $\omega > 1$.

Phase angle table : $G_1(j\omega) = \frac{12}{j\omega(1+j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}\omega$	ϕ_R
0.1	-90°	-5.71°	-95.71°
[illegible]	-90°	-45°	-135°
2	-90°	-63.43°	-153.43°
[illegible]	-90°	-90°	-180°

From the plot, ϕ_1 = P.M. = 15°, ω_{gc} = 3.5 rad/sec, G.M. = + ∞ dB

Step 3 : $\phi_s = 40^\circ$ (given) i.e. given P.M.

$$\therefore \quad \phi_m = \phi_s - \phi_1 + \varepsilon, \text{ Let } \varepsilon = 8^\circ$$

$$= 40^\circ - 15^\circ + 8^\circ = 33^\circ$$

Step 4 : $\sin \phi_m = \frac{1-\alpha}{1+\alpha} = \sin 33^\circ = 0.5446$

$$\therefore \quad 1 - \alpha = 0.5446\,(1+\alpha) \quad \text{i.e.} \quad \alpha = 0.2948$$

Choose $\alpha = 0.3$

Step 5 : $-10 \text{ Log}\left(\frac{1}{\alpha}\right) = -5.23 \text{ dB}$

Find the frequency from the Fig. 13.4.8 (a), which gain of uncompensated system is – 5.23 dB, which is ω_m.

$$\therefore \quad \omega_m = \mathbf{5.8 \text{ rad/sec}}$$

$$\text{But} \quad \omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$\therefore \quad T = 0.3147 \quad \text{i.e.} \quad \frac{1}{T} = 3.176$$

Step 6 : Two corner frequencies of lead compensator are,

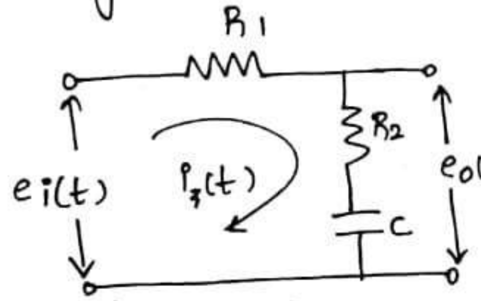
$$\omega_{C1} = \frac{1}{T} = 3.176 \quad \omega_{C2} = \frac{1}{\alpha T} = \frac{1}{0.0944} = 10.6$$

Step 7 : $K = K_c \alpha$

$$\therefore \quad K_c = \frac{K}{\alpha} = \frac{12}{0.3} = 40$$

LAG COMPENSATOR

consider an electrical Network which is a lag compensating Network,



Assuming unloaded circuit and applying KVL to the loop.

$$e_i(t) = i(t)R_1 + i(t)R_2 + \frac{1}{C}\int i(t)\,dt$$

Taking Laplace Transform,

$$E_i(S) = I(S)\left[R_1 + R_2 + \frac{1}{CS}\right] \longrightarrow ①$$

Now o/p equation is

$$e_o(t) = i(t)R_2 + \frac{1}{C}\int i(t)\,dt.$$

Taking Laplace Transform,

$$E_o(S) = I(S)\left[R_2 + \frac{1}{SC}\right] \longrightarrow ②$$

substitute I(S) from eq ② to ①.

$$E_i(S) = \frac{E_o(S)}{\left[R_2 + \frac{1}{SC}\right]}\left[R_1 + R_2 + \frac{1}{SC}\right]$$

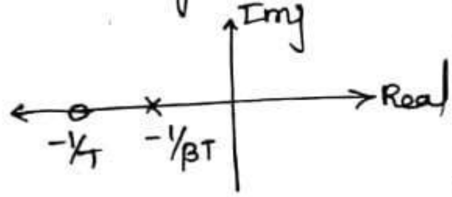
$$E_i(S) = \frac{E_o(S)\left[(R_1+R_2)SC + 1\right]}{1 + R_2 SC}$$

$$\frac{E_o(S)}{E_i(S)} = \frac{1 + SR_2C}{1 + S(R_1+R_2)C} = \left(\frac{R_2}{R_1+R_2}\right)\left[\frac{S + \frac{1}{R_2C}}{S + \frac{1}{(R_1+R_2)C}}\right]$$

This is generally expressed as,

$$\frac{E_o(s)}{E_i(s)} = \left(\frac{1}{\beta}\right)\frac{s+1/T}{s+1/\beta T} \quad \text{where } T = R_2 C,\ \beta = \frac{R_1+R_2}{R_2} > 1$$

The lag compensator has zero at $s = -1/T$ and pole at $s = -1/\beta T$. As $\beta > 1$, the pole is always located to the right of the zero.



Maximum lag angle & β :

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{\beta}\left(\frac{s+1/T}{s+1/\beta T}\right) = \frac{1+Ts}{1+\beta Ts}$$

In frequency domain, we get,

$$\frac{E_o(j\omega)}{E_i(j\omega)} = \frac{1+j\omega T}{1+j\omega\beta T}$$

$$M = \left|\frac{E_o(j\omega)}{E_i(j\omega)}\right| = \frac{\sqrt{1+\omega^2T^2}}{\sqrt{1+\omega^2T^2\beta^2}} \Rightarrow \text{Magnitude}$$

$$\boxed{\phi = \tan^{-1}\omega T - \tan^{-1}\omega\beta T} \Rightarrow \text{phase}$$

The equation is similar exactly to the lead network, only $\beta > 1$.

$$\frac{d\phi}{d\omega} = 0$$

$$\frac{d}{d\omega}\left[\tan^{-1}\omega T - \tan^{-1}\omega\beta T\right] = 0$$

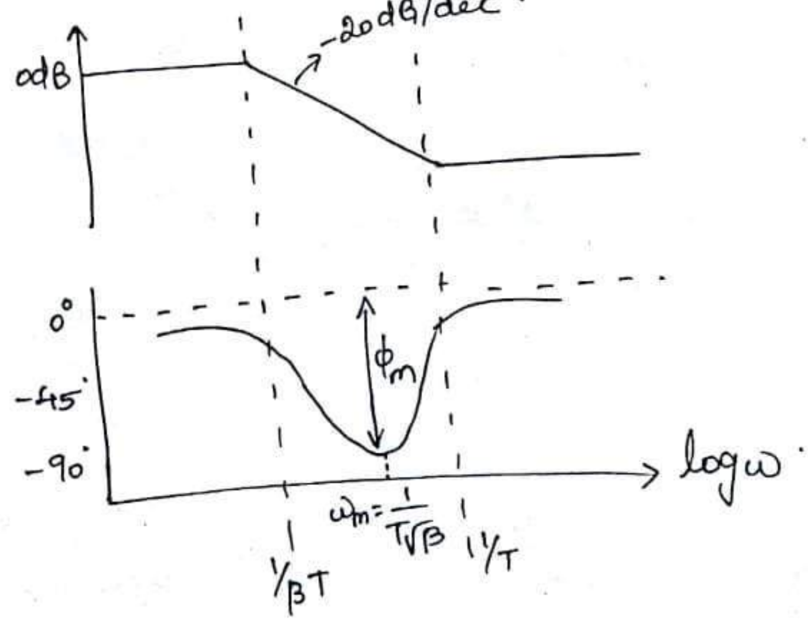
Solving we get,

$$\omega_m = \frac{1}{T\sqrt{B}} = \sqrt{\frac{1}{T}\left(\frac{1}{\beta T}\right)}$$

This is the frequency at which phase lag is at its maximum. The two corner frequencies of lag compensator are, $\boxed{\omega_{C1} = \frac{1}{T} \; ; \; \omega_{C2} = \frac{1}{\beta T}}$

Thus ω_m is geometric mean of two corner frequencies the phase lag angle does not play a role in the lag compensation.

Bode plot of Lag compensator:



$\omega_{C1} = \frac{1}{\beta T}$ for a pole at $s = -\frac{1}{\beta T}$

$\omega_{C2} = \frac{1}{T}$ for a zero at $s = -1/T$.

Steps to design lag compensator.

Step 1: Assume a lag compensator having Transfer function. $G_c(s) = \frac{1}{\beta} \frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})} = \frac{1+Ts}{1+\beta Ts}$

Assume $G_1(s) = K G(s)$.

From given error constant, determine the value of K which satisfies the steady state performance.

Step 2: using the value of K determined above, draw Bode plot of $G_1(j\omega)$ obtain the phase margin. This is say ϕ_1 for uncompensated system.

Step 3: let ϕ_s = P.M. specified

$\phi_2 = \phi_s + \varepsilon$

$\varepsilon \rightarrow$ margin of safety $= 5°$ to $15°$

Then ε compensates for phase lag of lag compensator.

Step 4: Find the frequency ω_2 corresponding to phase margin of ϕ_2 degrees. ie.,) the frequency at which phase angle of open loop T.F is $-180° + \phi_2$. Choose this new gain cross over frequency.

<u>Step5:-</u> To have ω_2 as new gain cross over frequency, determine the attenuation necessary to shift the magnitude curve up and down to 0dB. This shift is due to the contribution of β which is $20 \log 1/\beta$.

$\therefore$ shift to have ω_2 as new gain crossover

$$= 20 \log \frac{1}{\beta} = -20 \log \beta .$$

<u>Step6:</u> choose upper corner frequency '$\frac{1}{T}$' which is $\frac{1}{2}$ or $\frac{1}{10}$ below the ω_2 is determined in step4.

$$\omega_{c2} = \frac{1}{T} = \frac{\omega_2}{2} \text{ or } \frac{\omega_2}{10}.$$

Thus determine the value of T.

The other corner frequency for lag compensator is $\boxed{\omega_{c1} = \frac{1}{\beta T}}$

<u>Step7:</u> Thus once Transfer function of lag compensator is known, draw Bode plot of compensated System and check the specifications. If it is not satisfied, repeat the design by modifying pole zero locations of compensator till a satisfactory result is obtained.

Effects & limitations of lag compensator:-

1. It allows high gain at low frequencies. thus it basically low pass filter.

2. attenuation shift ω_{gc} to low freq. points. Thus Bandwidth gets reduced.

3. If B.W reduced, provides slower response. Thus rise time and settling time are usually longer. The transient response lasts for longer time.

4. S/m becomes more sensitive to parameter Variations.

5. It acts a PI controller. thus tends to make system less stable.

— × —

1. For a certain system, $G(s) = \frac{0.025}{s(1+0.5s)(1+0.05s)}$

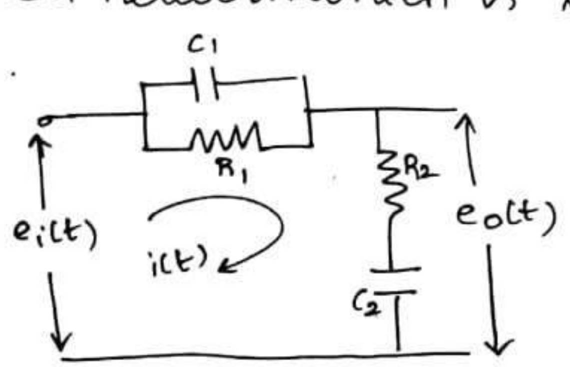
Design a suitable lag compensator to give, velocity error constant $= 20\ sec^{-1}$ and $P.M = 40^\circ$.

= Step 1: Assume $G_1(s) = K\,G(s)$

$$= \frac{0.025K}{s(1+0.5s)(1+0.05s)}$$

Lag-lead compensator:

A combination of lag and lead compensator is nothing but lag-lead compensator. consider an electrical network which is lag-lead network,



Let us obtain the transfer function of electrical lag-lead network. Now, sum of currents through R_1 and C_1 is nothing but $i(t)$.

$$\frac{e_i - e_o}{R_1} + C_1 \frac{d(e_i - e_o)}{dt} = i(t)$$

Taking Laplace Transform we get,

$$\frac{1}{R_1} E_i(s) - \frac{1}{R_1} E_o(s) + \frac{1}{sC_1} E_i(s) - sC_1 E_o(s) = I(s) \rightarrow (1)$$

The output equation is,

$$i(t) R_2 + \frac{1}{C_2} \int i(t)\, dt = e_o(t)$$

Taking Laplace Transform,

$$I(s)\left[R_2 + \frac{1}{sC_2}\right] = E_o(s) \rightarrow (2)$$

Substitute (1) in (2) we get,

$$\left\{E_i(s)\left[\frac{1}{R_1}+sC_1\right] - E_0(s)\left[\frac{1}{R_1}+sC_1\right]\right\}\left[R_2+\frac{1}{sC_2}\right] = E_0(s)$$

$$\left[\frac{E_i(s)(1+sR_1C_1) - E_0(s)(1+sR_1C_1)}{R_1}\right]\left[\frac{1+sR_2C_2}{sC_2}\right] = E_0(s)$$

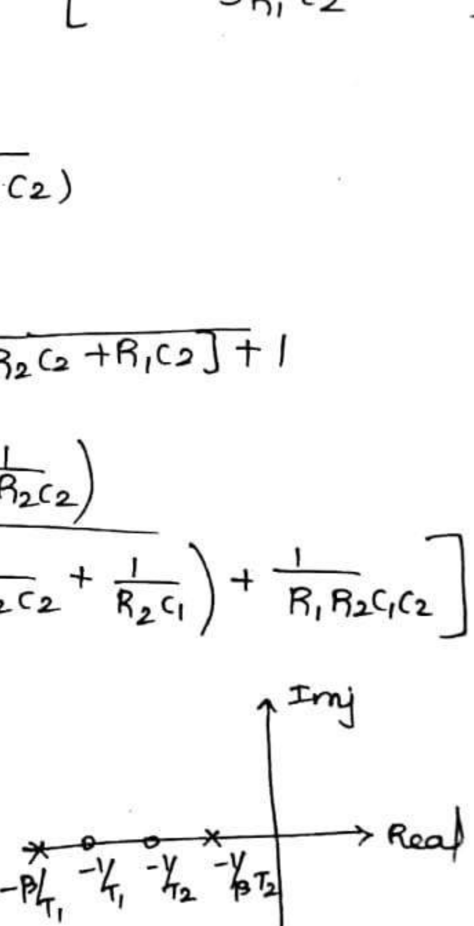
$$E_i(s)\frac{[(1+sR_1C_1)(1+sR_2C_2)]}{sR_1C_2} = E_0(s)\left[1+\frac{(1+sR_1C_1)(1+sR_2C_2)}{sR_1C_2}\right]$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1+sR_1C_1)(1+sR_2C_2)}{sR_1C_2 + (1+sR_1C_1)(1+sR_2C_2)}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(1+sR_1C_1)(1+sR_2C_2)}{s^2R_1R_2C_1C_2 + s[R_1C_1+R_2C_2+R_1C_2]+1}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{R_1R_2C_1C_2\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{R_1R_2C_1C_2\left[s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1R_2C_1C_2}\right]}$$

$$\frac{E_0(s)}{E_i(s)} = \frac{(s+1/T_1)(s+1/T_2)}{(s+\beta/T_1)(s+1/\beta T_2)}$$



where $T_1 = R_1C_1$; $T_2 = R_2C_2$.

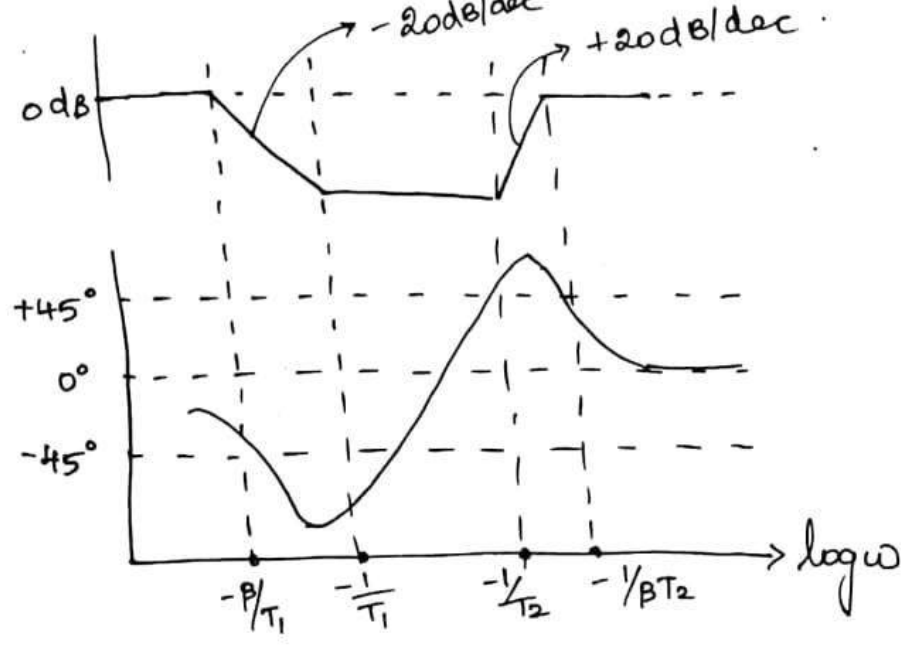
$$\frac{\beta}{T_1}+\frac{1}{\beta T_2} = \frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}$$

$$\alpha\beta T_1T_2 = R_1R_2C_1C_2$$

$\alpha\beta = 1$. It also can be expressed as,

$$\frac{E_0(s)}{E_i(s)} = \frac{(1+T_1s)(1+T_2s)}{\left(1+\frac{T_1}{\beta}s\right)(1+T_2\beta s)} \cdot \text{ where } \beta > 1$$

Bode plot of lag-lead compensator:



Effects of lag-lead compensator:

It is used when both fast response and good static accuracy are desired. use of lag-lead compensator increases the low frequency gain which improves the steady state. while at same time, it increases Bandwidth of the s/m, making system response very fast.

— x —— x —

Design of lag lead compensator is illustrated with an example.

1. Consider unity feedback system whose OLTF is $G(S)=\frac{K}{S(S+1)(S+2)}$ Design suitable lag-lead compensator. so as to achieve static velocity

error constant = 10 sec^{-1}, phase margin = 50°, gain margin $\geq$ 10dB.

= Solution:-

Transfer function of compensator is,

$$G_c(s) = \frac{(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)(1+\beta T_2 s)}$$

$$K_v = \lim_{s\to 0} s\, G_c(s)\, G(s)$$

$$10 = \lim_{s\to 0} \frac{\cancel{s}(1+T_1 s)(1+T_2 s)}{\left(1+\frac{T_1}{\beta}s\right)\left(1+\beta T_2 s\right)} \cdot \frac{K}{\cancel{s}(s+1)(s+2)}$$

$$10 = \frac{K}{2} \qquad \therefore \boxed{K = 20}$$

uncompensated system, $G_1(s) = \dfrac{20}{s(s+1)(s+2)}$

$$G_1(s) = \frac{10}{s(1+s)(1+0.5s)}$$

Draw Bode plot for uncompensated s/m.

Factors: $20 \log 10 = 20$dB.

1 pole at origin, -20 dB/dec.

$\omega_{c1} = 1$, simple pole, -20dB/dec

$\omega_{c2} = 2$, simple pole, -20dB/dec.

For stability of system K should be >0

$$\therefore K > 0$$

$$\frac{0.65 - 0.1K}{0.65} > 0$$

$$\frac{0.65}{0.1} > K$$

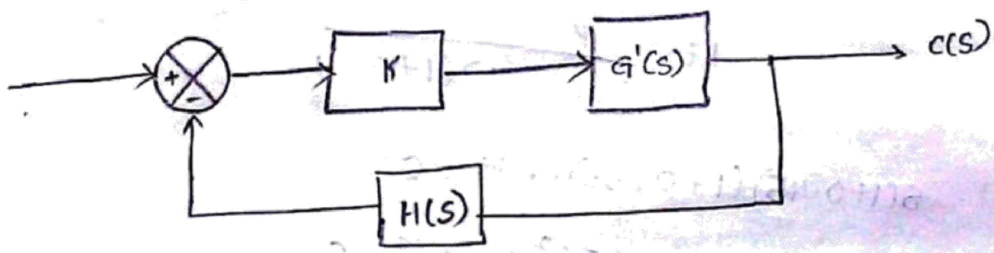
$$0 < K < 6.5$$

* Root Locus :

Movement of closed loop poles with small changes in the parameters of the system can be known by Root Locus method. This method is introduced by W.R. Evans. in 1948.

This is a graphical method in which movement of poles in the s-plane was sketched when a parameter of system is varied from 0 to ∞

* Concept of Root Locus :



Characteristic Eqn

$$1 + G(s)H(s) = 0.$$

where $G(s) = KG'(s)$

$$\Rightarrow 1 + KG'(s)H(s) = 0.$$

The roots of the above equation are now dependent upon K. If K is varied from $-\infty$ to $+\infty$

then for each value of K, we will get seperate set of locations of the roots of characteristic equation, If all such locations are joined, the resulting locus is called Root locus.

When K is varied from 0 to ∞, the plot is called 'Direct Root Locus' and when K is varied from −∞ to ∞, the plot obtained is 'Inverse Root Locus'.

Ex:

consider unity feedback system with $G(s) = \frac{K}{s}$. Obtain the Root Locus.

we have $1 + G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K}{s} = 0$$

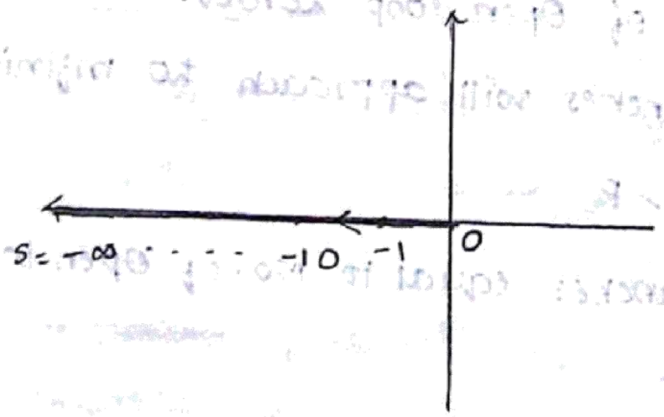
$$\Rightarrow K + s = 0.$$

$$\Rightarrow s = -K$$

For $K = 0$, $s = 0$, for $K = 100$, $s = -100$

$K = 1$, $s = -1$, ⋮

$K = 10$, $s = -10$, $K = \infty$, $s = -\infty$.



* Rules of constructing a Root Locus :

1. The Root Locus is always symmetrical about real axis.

2. The Roots of characteristic Equation are either Real or complex conjugate or combination of both. Therefore their locus must be symmetrical about Real axis of the s-plane.

2. Let $G(s) H(s)$ = open loop transfer function of the system where 'P' be no. of open loop poles and Z = no. of open loop zeroes.

case (i) : $P > Z$

i) the no. of branches 'N' equal to no. of open loop poles i.e., $N = P$.

ii) Branches will start from each of the location of open loop pole. out of 'P' no. of branches 'Z' no. of branches will terminate at the locations of Open loop zeroes. Remaining '(P-Z)' branches will approach to infinity (∞)

case (ii) : $Z > P$

i) No. of Branches equal to no. of Open loop zeroes i.e., $N = Z$.

ii) The Branches will terminate at each of the finite location of open loop zero.. But out start from each of the finite open loop pole, while remaining '(z-p)' no. of branches will originate from infinity (∞) and will approach to finite zeroes.

case (iii) : $P = Z$

i) The no. of branches $N = P = Z$, seperate Branch will start from each of the open loop pole and terminate at open loop zero. No Branch will start or terminate at ∞ when $P = Z$.

3. A point on the Real axis lies on the root locus if the sum of number of open loop poles and zeroes on the real axis, to the right hand side of this point is odd.

4. The $(n-m)$ root locus branches that tend to ∞, along the straight lines are called 'Asymptotes'. Angle of such Asymptotes are given by

$$\phi_A = \frac{\pm 180^\circ [2q+1]}{(n-m)}$$

where $q = 0, 1, 2, \ldots (n-m)-1$

m = no. of zeroes.

n = no. of poles.

Asympotes are always symmetrical about Real axis.

5. The point of Intersection of Asymptotes with the

6. The point from where branches break into complex from real is called Break-away point. The point from where branches break into Real from complex is called Break-in point.

The Break-away and Break-in points of the Root Locus are determined from Roots of the Equation $dK/ds = 0$

If 'r' no. of branches of Root Locus meet at a point, then they break-away at an angle of $\pm \frac{180^\circ}{r}$.

7. The angle of departure from a complex open loop pole is given by $\phi_p = \pm 180^\circ(2q+1) + \phi$

where $q = 0, 1, 2, \ldots$,

ϕ = net angle contribution at the pole by all other open loop poles and zeroes.

The angle of arrival at a complex open loop zero is given by $\phi_z = \pm 180^\circ(2q+1) + \phi$

where ϕ = net angle contribution at the zero by all other open loop poles and zeroes.

8. The Intersection of Root Locus branches with an Imaginary axis can be determined by use of Routh criterion (or) by letting $s = j\omega$ in the characteristic equation and equating Real part and Imaginary part to zero, to solve for

point on imaginary axis and K is the value of gain at the intersection point.

9. Open Loop Gain $K = \dfrac{\text{Product of vector lengths from open loop poles to the point } s_a}{\text{Product of vector lengths from open loop zeroes to the point } s_a}$.

1. A unity feedback control system has an open loop transfer function $G(s)H(s) = \dfrac{K}{s(s^2+4s+13)}$. sketch the root Locus.

sol:- Step-1 :

Locate the poles and zeroes of G(s).H(s) on the s-plane. The Root Locus branch starts from open loop poles to terminates at zeroes.

To locate poles and zeroes,

$$s(s^2+4s+13) = 0$$

$$s=0, \quad s^2+4s+13=0$$

$$s = \frac{-4 \pm \sqrt{16-4(13)}}{2}$$

$$s = -2 \pm \sqrt{4-13}$$

$$s=0, \; s = -2 \pm j3$$

The number of root locus branches is equal to no. of poles of open loop transfer function.

Step-2 :

Determine the Root locus on the real axis.

Step-3 :

To find angle of asymptots and centroid and meeting point of asymptots with real axis.

branches going to ∞.

Angle of Asymptotes $= \frac{\pm 180^\circ(2q+1)}{n-m}$

$q = 0, 1, 2, \ldots [(n-m)-1]$

n = no. of poles = 3

m = no. of zeroes = 0.

$\Rightarrow q = 0, 1, 2$

$\Rightarrow$ Angles $= \pm \frac{180^\circ}{3} = \pm 60^\circ$, $(q=0)$

$= \pm \frac{180(3)}{3} = \pm 180^\circ$, $(q=1)$

$= \frac{\pm 180(5)}{3} = \pm 60(5) \pm 360^\circ$, $(q=2)$

$= \mp 60^\circ$

centroid $= \frac{\text{sum of poles} - \text{sum of zeroes}}{n-m}$

$= \frac{0 - 2 + j3 - 2 - j3}{3-0} = \frac{-4}{3} = -1.33$

Step-4 :

Find the break away & break in points.

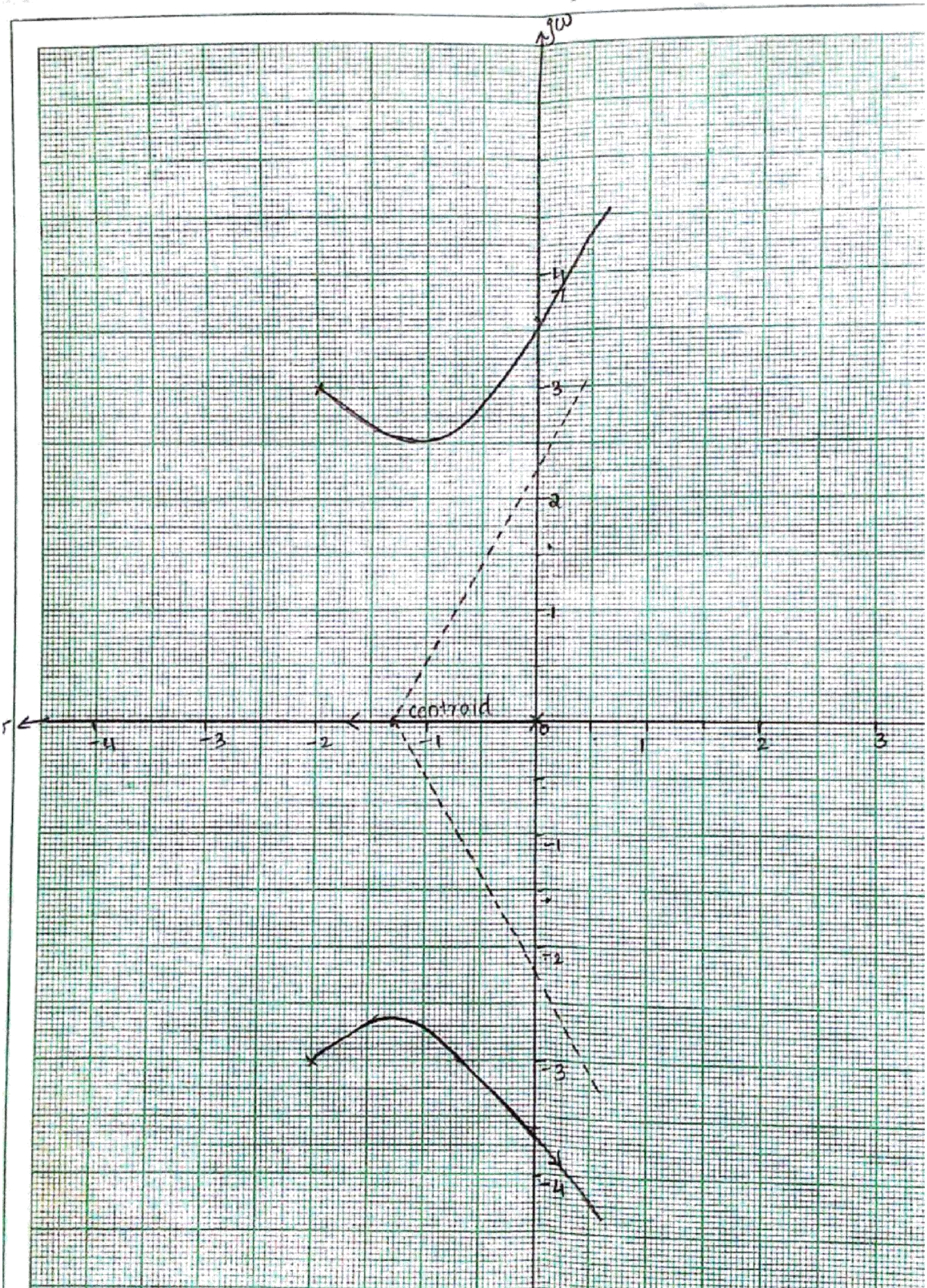
These points either lie on real axis or exist as complex conjugate pairs. If there is a root locus on real axis b/w 2 poles, then there exists a Break-away point.

Similarly, if there is a root locus on real axis b/w 2 zeroes, then there exists a Break-in point and if there is a root locus on real axis b/w

... then there may or may not be

branches going to ∞.

Example-1 : $G(s)\,H(s) = \dfrac{\quad}{s(s^2+4s+13)}$



If $B(s) + K A(s) = 0$ is characteristic Equation.

$K = -\frac{B(s)}{A(s)}$ then K should be real & positive then only we can say Breakaway & Breakin point exists.

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1+\frac{K}{s(s^2+4s+13)}}$$

$$\frac{G(s)}{1+G(s)H(s)} = \frac{K}{s(s^2+4s+13)+K}$$

characteristic Eqn $\quad s(s^2+4s+13)+K = 0$

$$\Rightarrow K = -s(s^2+4s+13)$$

$$\Rightarrow K = -s^3 - 4s^2 + 13s$$

$$\Rightarrow \frac{dK}{ds} = -3s^2 - 8s - 13$$

$$\frac{dK}{ds} = -(3s^2+8s+13) = 0$$

$$\Rightarrow 3s^2+8s+13 = 0$$

$$s = \frac{-8 \pm \sqrt{64-4(13)(3)}}{2(3)}$$

$$s = \frac{-4 \pm \sqrt{16-39}}{3}$$

$$s = \frac{-4}{3} \pm \frac{j\sqrt{23}}{3}$$

$$s = -1.33 \pm j1.6$$

$$\Rightarrow K = -s^3 - 4s^2 - 13s$$

$$K = -(-1.33 + j1.6)^3 - 4(-1.33 + j1.6)^2 - 13(-1.33 \pm j1.6)$$

K is neither positive nor real; hence Breakin
n Breakaway points does not exist.

Step-5 :

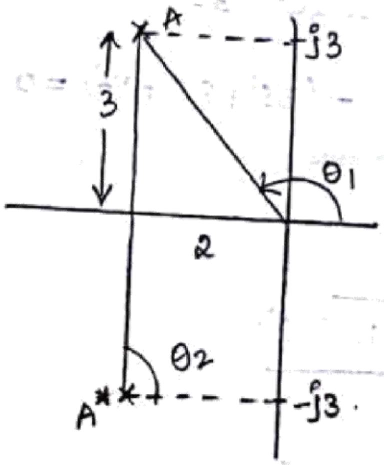
If there is a complex pole, then determine the angle of departure from the complex pole

If ther is a complex zero, then determine the angle of arrival from the complex zero.

Angle of departure =

180° - (sum of angles of vector to the complex pole A from other poles) + (sum of angles of vector to the complex poles A to zeroes)

$\phi_d = 180° - \phi$, $\phi = \Sigma\phi_p - \Sigma\phi_z$



$\theta_1 = 180° - \tan^{-1}\left(\frac{3}{2}\right) = 123.7°)$

$\theta_2 = 90°$

$\phi_d = 180° - (\theta_1 + \theta_2)$

(Angle of departure at A) $= 180° - (123.7° + 90°)$
$= 180° - 213.7°$
$= -33.7°$.

Angle of departure At $A^* = 33.7°$.

Step-6 :

Find the points where the Root Locii may cross the imaginary axis by RH criterion.

Characteristic eqn $s^3 + 4s^2 + 13s + K = 0$

s^3	1	13
s^2	4	K
s	$\frac{52-K}{4}$	0
s^0	K	

$\frac{52-K}{4} > 0$.

$K = 52$.

Auxiliary Eqn $4s^2 + K = 0$.

$4s^2 + 52 = 0$

$s^2 = -\frac{52}{4}$

$s = \pm j\sqrt{13}$

$s = \pm j3.6$

2. $G(s) = \frac{K(s+9)}{s(s^2+4s+11)}$

Sol:- step-1 :

Locate the poles and zeroes of G(s) on the s-plane. The Root locus branch starts from open loop poles and terminates at zeroes.

$s(s^2+4s+11) = 0$

$s = 0$, $s = \frac{-4 \pm \sqrt{16-4(11)}}{2}$

$s = -2 \pm \sqrt{4-11}$

$s = -2 \pm j\sqrt{7}$.

poles : $s=0$, $s = -2 \pm j2.6$.

zeroes : $s+9=0$, $s = -9$.

The no. of root locus branches is equal to no of poles of open loop poles.

step-2 :

Determine the root locus on the real axis.

step-3 :

Angle of Asymptotes $= \frac{\pm 180^\circ(2q+1)}{n-m}$

Angles $= \pm 90^\circ$ to $\pm 180^\circ \cdot 270^\circ$

$$\text{centroid} = \frac{\text{sum of poles} - \text{sum of zeroes}}{n-m}$$

$$= \frac{0-2+j2.6-2-j2.6+9}{2}$$

$$= \frac{-4+9}{2} = \frac{5}{2} = 2.5.$$

step-4 :

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K(s+9)}{s(s^2+4s+11)}}{1+\frac{K(s+9)}{s(s^2+4s+11)}}$$

$$= \frac{K(s+9)}{s(s^2+4s+11)+K(s+9)}$$

characteristic eqn :

$$s(s^2+4s+11)+K(s+9)=0.$$

$$K = \frac{-s(s^2+4s+11)}{(s+9)}$$

$$\frac{dK}{ds} = K = \frac{-s^3+4s^2-11s}{s+9}$$

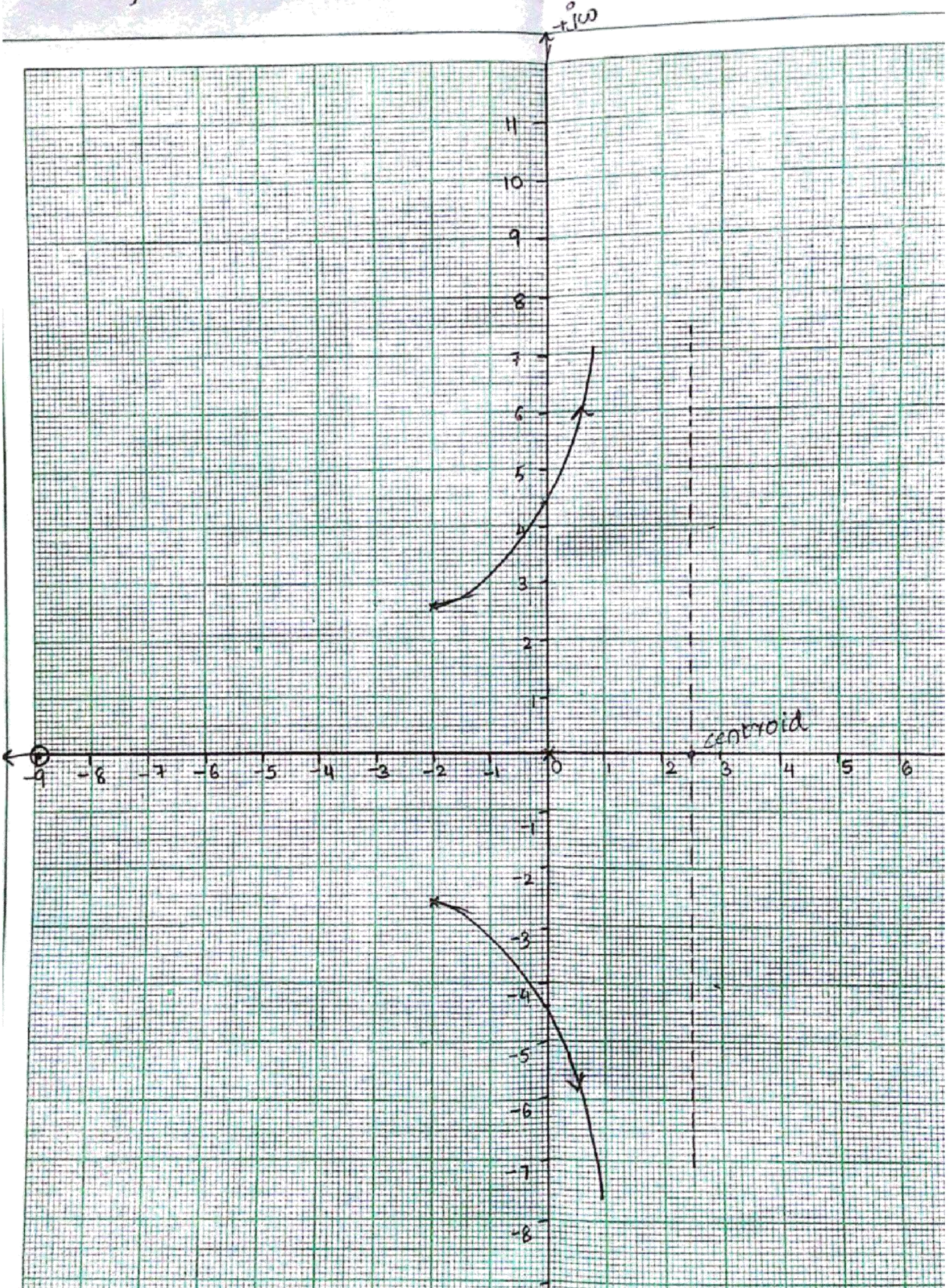
$$\frac{dK}{ds} = \frac{(s+9)[-3s^2-8s-11]-[-s^3-4s^2-11s](1)}{(s+9)^2} = 0.$$

$$\Rightarrow -(3s^2+8s+11)(s+9)+s^3+4s^2+11s=0$$

$$\Rightarrow s^3-3s^3-27s^2-8s^2-72s+\cancel{11s}-99+$$
$$s^3+4s^2+\cancel{11s}=0.$$

Angles $= \pm 90^\circ$ to $\pm 180^\circ \cdot 270^\circ$

Example - 2 :



$$\Rightarrow \quad 2s^3 + 31s^2 + 72s + 99 = 0.$$

$$s = -13.02, \quad s = -1.2 + 1.5j, \quad s = -1.2 - 1.5j$$

$$K = -\frac{s(s^2+4s+11)}{s+9}.$$

$$= \frac{-13.02\left[(-13.02)^2 + 4(-13.02) + 11\right)}{-13.02 + 9}$$

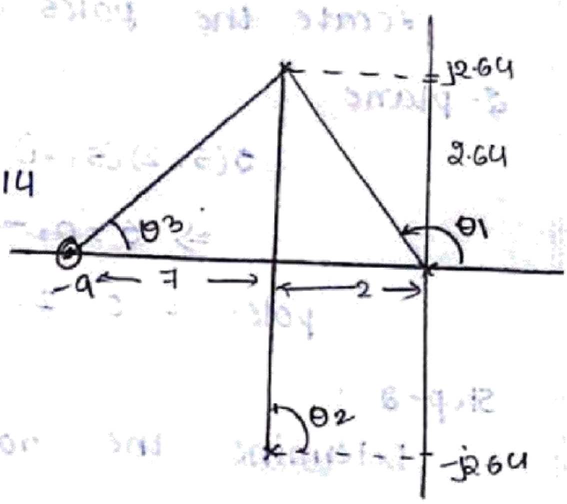
$$= -753.$$

$\because$ K is not positive, Break-away & Break-in pts does not exist.

Step-5 :

Angle of Departure

$$\phi_d = 180^\circ - (\theta_1 + \theta_2) + \theta_3$$



$$\theta_1 = 180^\circ - \tan^{-1}\left[\frac{2.64}{2}\right] = 127.14$$

$$\theta_2 = 90^\circ.$$

$$\theta_3 = \cancel{180^\circ -} \tan^{-1}\left[\frac{2.64}{7}\right]$$

$$= \cancel{159.38}\ 20.66.$$

$$\phi_d = 180^\circ - (127.14 + 90^\circ) + \cancel{159.38}\ 20.66$$

$$\phi_d = -16.48^\circ \text{ at } A, \quad A^* = 16.48^\circ.$$

Step-6 :

$$s^3 + 4s^2 + 11s + Ks + 9K = 0$$

$$s^3 + 4s^2 + s(11+K) + 9K = 0$$

s^3	1	11+K
s^2	4	9K

$$\frac{4(11+K) - 9K}{4} > 0.$$

$$44 + 4K - 9K > 0$$

$$4s^2 + 9K = 0$$

$$4s^2 + 9(8.8) = 0.$$

$$4s^2 = -79.2$$

$$s^2 = -19.8$$

$$s = \pm j4.4.$$

3. Sketch the root locus of the system whose open loop transfer function is $G(s) = \frac{K}{s(s+2)(s+4)}$. Find the value of K so that the damping ratio of closed loop system is 0.5. System is unity feedback system.

Sol:- Step-1 :

Locate the poles and zeroes of G(s) on the s-plane

$$s(s+2)(s+4) = 0.$$

$$\Rightarrow s = 0, -2, -4.$$

poles = 0, −2, −4

Step-2 :

Determine the root locus on the real axis.

Step-3 :

$$\text{Angle of Asymptots} = \pm \frac{180^\circ (2q+1)}{n-m}$$

$$n = 3,\ m = 0$$

$$q = 0, 1, \ldots [(n-m)-1]$$

$$q = 0, 1, 2.$$

for $q = 0$, $\text{angle} = \pm \frac{180^\circ(1)}{3} = \pm 60^\circ.$

$q = 1$, $\text{angle} = \pm \frac{180^\circ(3)}{3} = \pm 180^\circ$

$q = 2$, $\text{angle} = \pm \frac{180^\circ(5)}{} = \pm 300^\circ.$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeroes}}{(n-m)}$$

$$= \frac{0-2-4-0}{3}$$

$$= -\frac{6}{3} = -2.$$

Step-4 :

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}}.$$

$$= \frac{K}{s(s+2)(s+4)+K}$$

Charac. Eqn

$$s(s+2)(s+4)+K = 0.$$

$$\Rightarrow K = -(s^2+2s)(s+4)$$

$$\Rightarrow K = -(s^3+4s^2+2s^2+8s).$$

$$\Rightarrow K = -(s^3+6s^2+8s)$$

$$\frac{dK}{ds} = -3s^2-12s-8.$$

$$\frac{dK}{ds} = 0$$

$$\Rightarrow 3s^2+12s+8 = 0$$

$$s = \frac{-12 \pm \sqrt{144-4(24)}}{2(3)}$$

$$s = \frac{-8 \pm 2\sqrt{10}}{2(3)}$$

$$s = -\frac{8}{3} \pm \frac{2\sqrt{10}}{3}$$

$$\cancel{s = -0.55, -4.77}$$

$$s = -0.84, -3.154.$$

$s = -0.84$, $K = -(s^3+6s^2+8s)$

$$= \cancel{-[(-0.55)^3} + 6(-0.55)^2 + 8(-0.55))$$

$$[(-0.84)^3 + 6(-0.84)^2 + 8(-0.84))$$

for $s = -3.154$

$K = -3.07$.

for $s = -0.84$, K is real and positive.

$\because$ there are no complex roots, angle of departure is not required.

Step-5:

$$s^3 + 6s^2 + 8s + K = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \frac{48-K}{6} & 0 \\ s^0 & K & \end{array}$$

$\frac{48-K}{6} > 0$.

$48 - K > 0$

$K < 48$.

Auxiliary Eqn $6s^2 + K = 0$.

$6s^2 + 48 = 0$

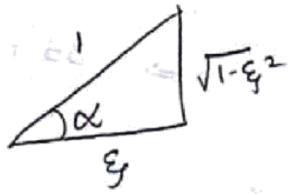
$s^2 = -\frac{48}{6}$

$s^2 = -8$.

$s = \pm j2.828$

To find the value of K corresponding to $\xi = 0.5$.

$\cos\alpha = \frac{\xi}{1}$



$\Rightarrow \alpha = \cos^{-1}(\xi)$

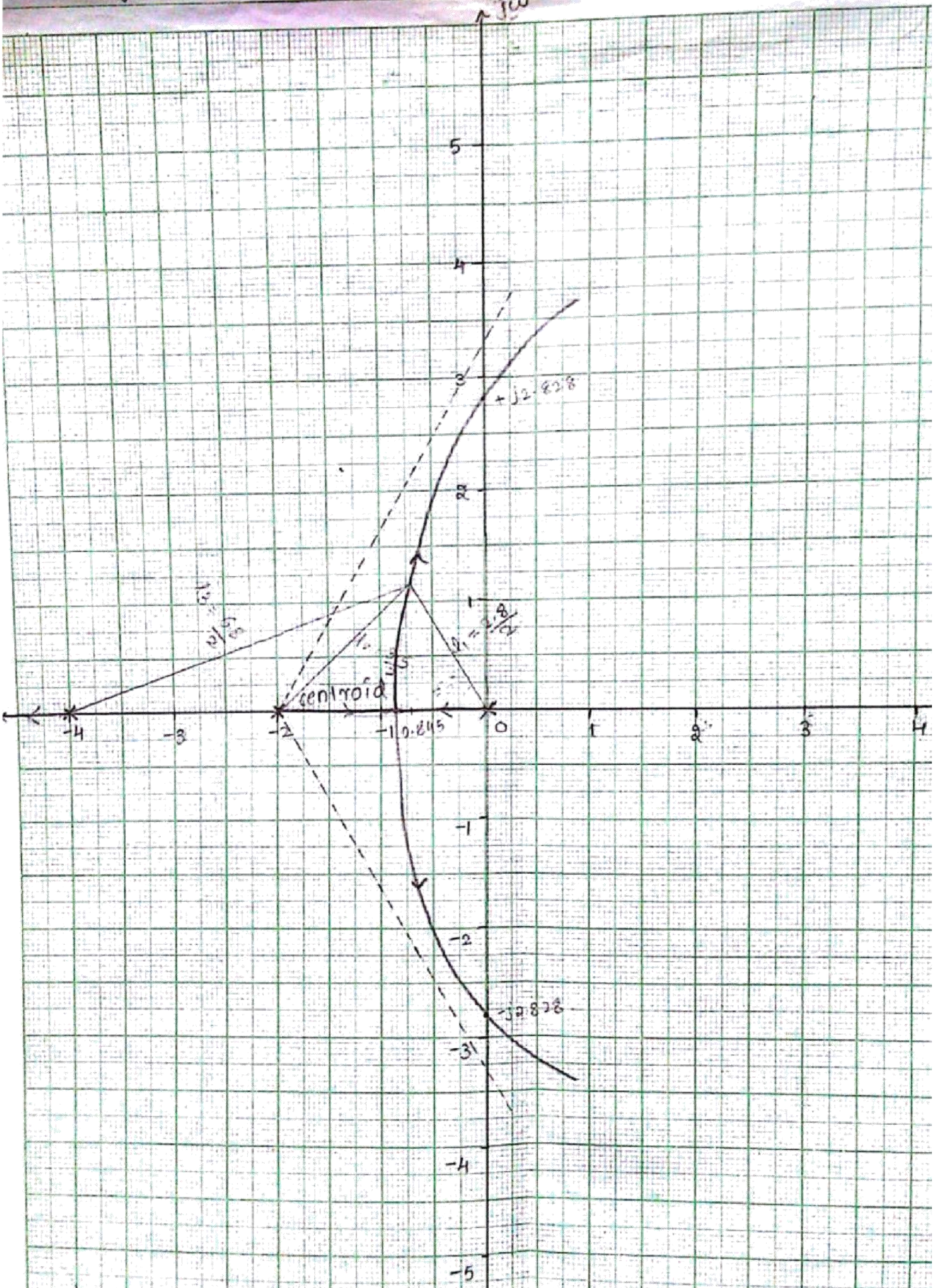
$= \cos^{-1}(0.5)$

$= 60°$.

Open loop gain K = $\dfrac{\text{Product of vector lengths from open loop poles to point } s_a}{\text{Product of vector lengths from open loop zeroes to point } s_a}$.

$l_1 = \frac{2.8}{8} = 1.4$, $l_2 = \frac{3.5}{...} = 1.75$, $l_3 = 6.8 - 3.4$

Example - 3 :



Complete root locus has 3 branches.

one branch starts at a pole at $s=-4$ and travel through -ve real axis to meet the zero at ∞.

The other two root locus branches starts at $s=0$ & $s=-2$ and travel through -ve real axis, break away from real axis at $s=-0.845$, then crosses imaginary axis at $s=\pm j2.8$ and travel parallel to Asymptots to meet the zeroes at ∞.

4. The open loop transfer function of unity feedback control system is given by $G(s) = \frac{K(s+1)}{s^2}$, sketch the root locus.

Sol:- Step-1 :

poles : $s^2 = 0$

$\Rightarrow s = 0, 0$

zeroes : $s+1 = 0$

$s = -1$.

Locate poles and zeroes on s-plane.

Step-2 :

Determine the root locus on the real axis.

Step-3 :

Angle of Asymptotes $= \frac{\pm 180^\circ(2q+1)}{n-m}$

$n = 2, \quad m = 1$

$\Rightarrow q = 0, 1, 2, \ldots [(n-m)-1]$

$= 0, 1, 2, \ldots [(2-1)-1]$

$q = 0$

Angle for $q=0$ $= \frac{\pm 180^\circ(1)}{1}$

$= \pm 180^\circ$.

sum of poles - sum of zeroes

step-4 :

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K(s+1)}{s^2}}{1+\frac{K(s+1)}{s^2}} = \frac{K(s+1)}{s^2+K(s+1)}$$

$$s^2+K(s+1)=0.$$

$$s^2+Ks+K=0.$$

$$K=\frac{-s^2}{s+1}.$$

$$\frac{dK}{ds}=0$$

$$\Rightarrow \frac{(s+1)(-2s)-(-s^2)(1)}{(s+1)^2}=0$$

$$\Rightarrow -2s(s+1)+s^2=0$$

$$\Rightarrow -2s^2-2s+s^2=0.$$

$$\Rightarrow -s^2-2s=0$$

$$\Rightarrow s^2+2s=0$$

$$\Rightarrow s=0,-2.$$

for $s=0$, $K=0$.

for $s=-2$, $K=\frac{-4}{-2+1}=4$

for $s=-2$, 0,K is real & positive.

∴ there are no complex roots, angle of departure does not exist

$$s^2+Ks+K=0$$

s^2	1	K
s^1	K	0
s^0	$\frac{K^2}{K}=K$	

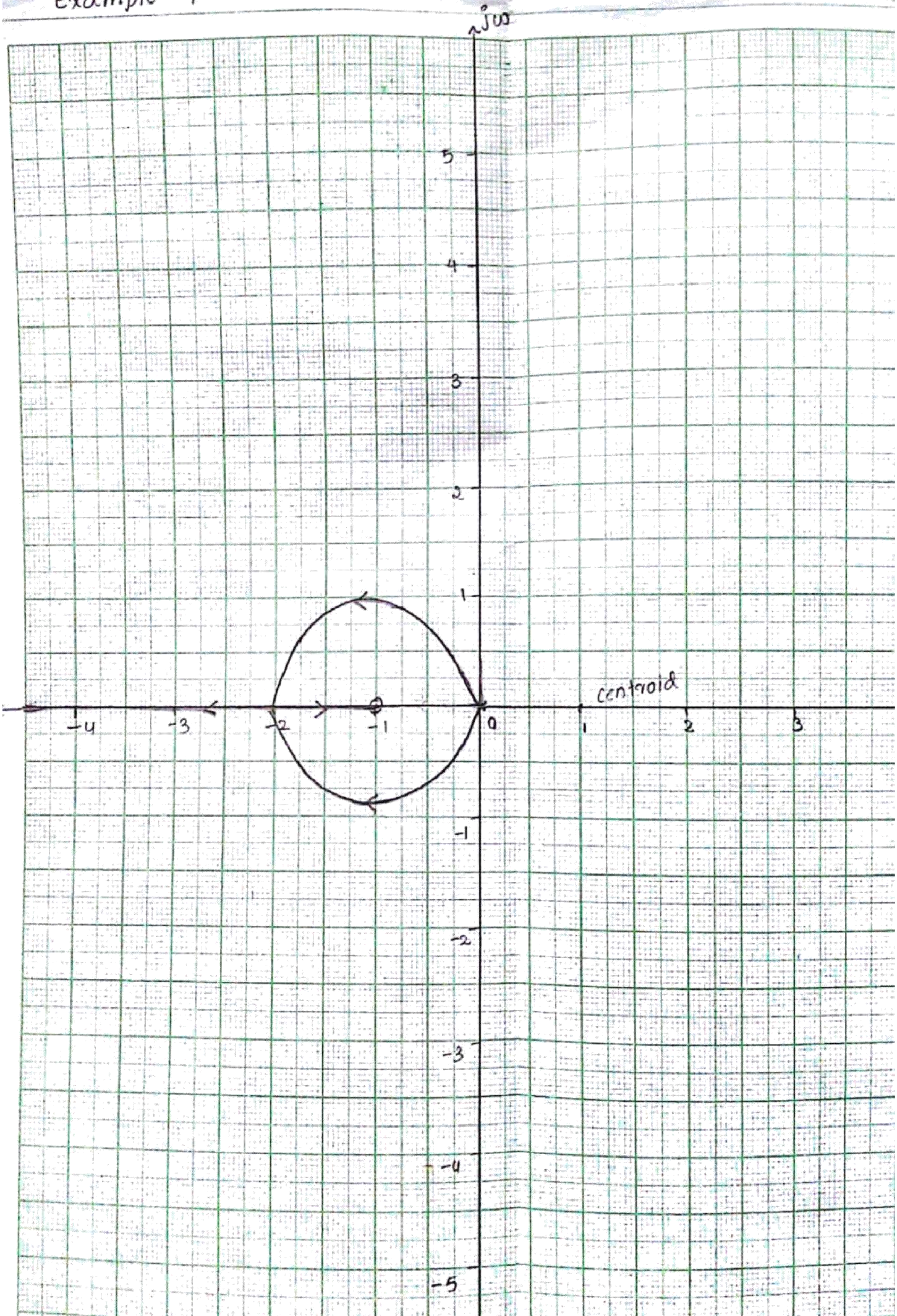
$K>0$.

$Ks=0$

$s=0$.

$K(s+1)$

Example - 4 :



* Effect of addition of open loop poles and zeroes:

Addition of poles:

1. Root locus shifts towards imaginary axis.
2. System stability relatively decreases.
3. System becomes more oscillatory in nature.
4. Range of operating values of K for stability of the system decreases.

Addition of zeroes:

1. Root Locus shifts to left away from imaginary axis.
2. Relative stability of the system increases
3. System becomes less oscillatory
4. Range of operating values of K for system stability increases.

Sol:-

$$G(s)\,H(s) = \frac{K(s+1)}{s(s-1)(s^2+4s+16)}$$

poles : $s(s-1)(s^2+4s+16) = 0.$

$s=0,\ s=1,\quad s^2+4s+16=0$

$s = -2 \pm 2\sqrt{3}j$

$s = -2 \pm 3.46j$

zeroes : $s+1=0$

$s=-1.$

Step 2 :

Determine the root locus on the real axis

Step 3 :

$$\text{Angle of Asymptotes} = \pm\frac{180^\circ(2q+1)}{n-m}$$

$n=4,\quad m=1$

$q = 0, 1, 2 \quad (n-m)-1$

$(4-1)-1.$

for $q=0$, angle $= \pm \frac{180^\circ(1)}{4-1} = \pm 60^\circ$,

for $q=1$, $\pm \frac{180^\circ(3)}{3} = \pm 180^\circ$

for $q=2$, $\pm \frac{180^\circ(5)}{3} = \pm 300^\circ = \mp 60^\circ$.

$$\text{centroid} = \frac{\text{sum of poles} - \text{sum of zeroes}}{n-m}$$

$$= \frac{0+1-2-j\cancel{3.4}-2+j\cancel{3.6}+1}{3}$$

$$= \frac{-2}{3} = -0.66.$$

step-4 :

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{K(s+1)}{s(s-1)(s^2+4s+16)}}{1+\frac{K(s+1)}{s(s-1)(s^2+4s+16)}}$$

$$= \frac{K(s+1)}{s(s-1)(s^2+4s+16)+K(s+1)}.$$

$$s(s-1)(s^2+4s+16)+K(s+1)=0$$

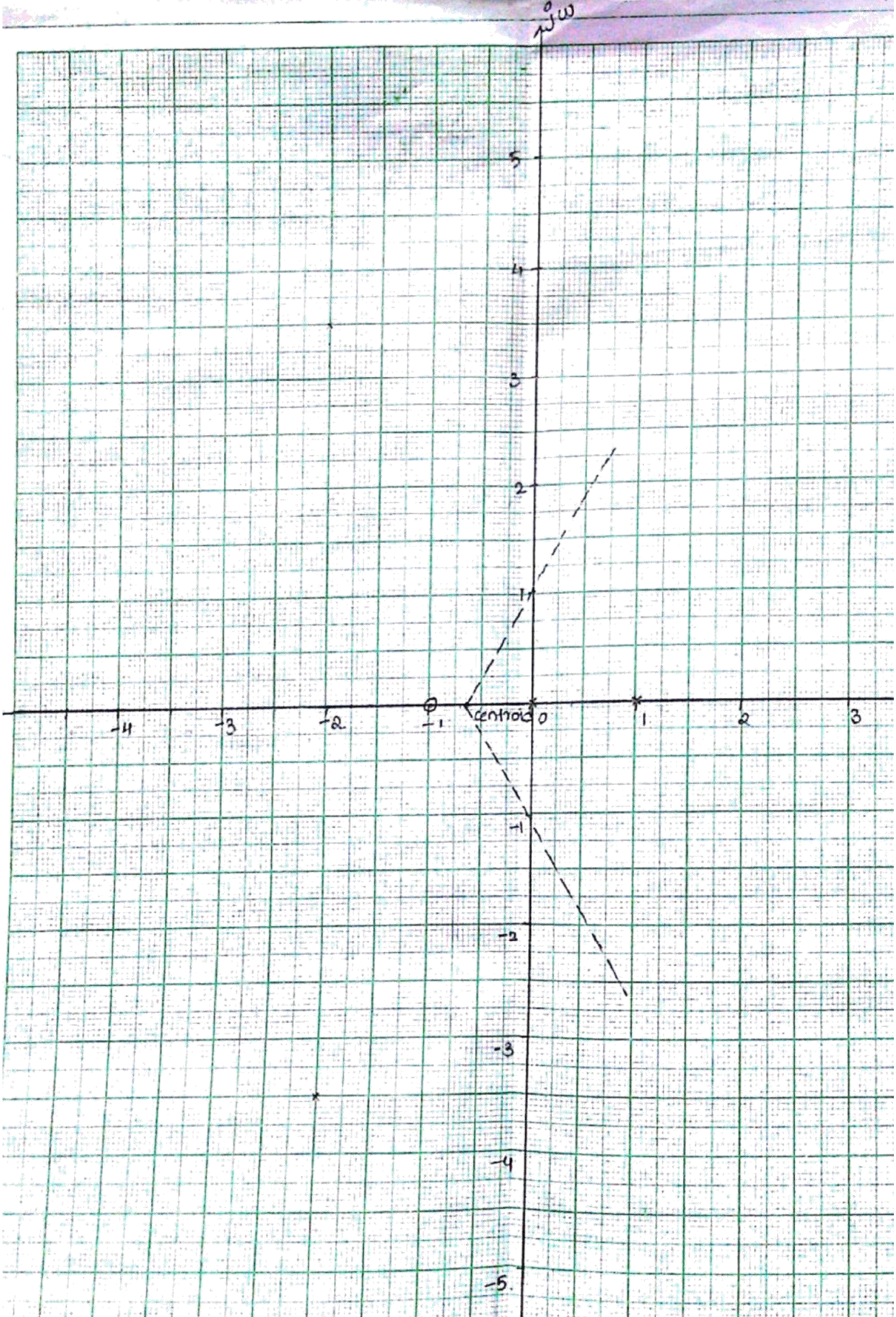
$$K = \frac{-(s^2-s)(s^2+4s+16)}{(s+1)}$$

$$K = \frac{-[s^4+4s^3+16s^2-s^3-4s^2-16s]}{s+1}$$

$$K = -\frac{(s^4+3s^3+12s^2-16s)}{s+1}.$$

$$\frac{dK}{ds}=0 \Rightarrow \frac{(s+1)[-(4s^3+9s^2+24s-16)]+(s^4+3s^3+12s^2-16s)}{(s+1)^2}=0$$

$\pm 180°(1) = \pm 60°$



$$\Rightarrow 4s^4 + 6s^3 + 24s^2 - 16s + 4s^3 + 6s^2 + 24s - 16 + s^4 + 3s^3 + 12s^2 - 16s = 0.$$

$$\Rightarrow \quad 5s^4 + 13s^3 + 42s^2 - 8s - 16 = 0.$$

$$\Rightarrow 4s^4 + 9s^3 + 24s^2 - 16s + 4s^3 + 9s^2 + 24s - 16 - s^4 - 3s^3 - 12s^2 + 16s = 0$$

$$\Rightarrow \quad 3s^4 + 10s^3 + 21s^2 + 24s - 16 = 0$$

STATE SPACE ANALYSIS OF CONTINUOUS SYSTEMS

Basically, there are 2 approaches to the analysis & Design of control systems. 1. The Transfer function approach. 2. State variable Approach.

Transfer function Approach	State variable Approach
1. Transfer function Approach is also called conventional Approach (or) Classical Approach	1. State variable Approach is called modern Approach
2. It is applicable to linear-Time Invariant systems, It is generally limited to single input single o/p systems.	2. It is applicable to linear as well as non-linear, Time variant as well as Time invariant, single i/p single o/p as well as multi i/p multi o/p systems.
3. In this initial conditions are neglected.	3. They are considered.
4. It is basically a frequency domain Approach.	4. It is a time domain Approach.
5. It is based on Trial & Error procedure	5. It is not based on Trial & Error procedure.
6. Only i/p, o/p & Error signals are considered important. The i/p & o/p variables must be measurable.	6. ~~The i/p & o/p variables~~ state variable need not represent physical variables. They need not be measurable & observable.

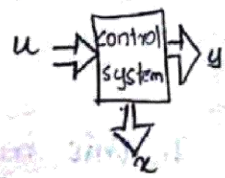
8. Transfer function of a system is unique	8. State model of a system is not unique

* Concepts of State, state variable and state Model :

State : The state of a dynamic system is a minimal set of variables such that the knowledge of these variables at t=to together with the knowledge of inputs for $t \geq t_0$ completely determines the behaviour of the system for $t \geq t_0$

State variable :

Set of variables which describe the system at any time instant are called state variables. In the state variable formulation of a system, in general, a system consists of 'm' inputs, 'p' outputs, 'n' state variables.



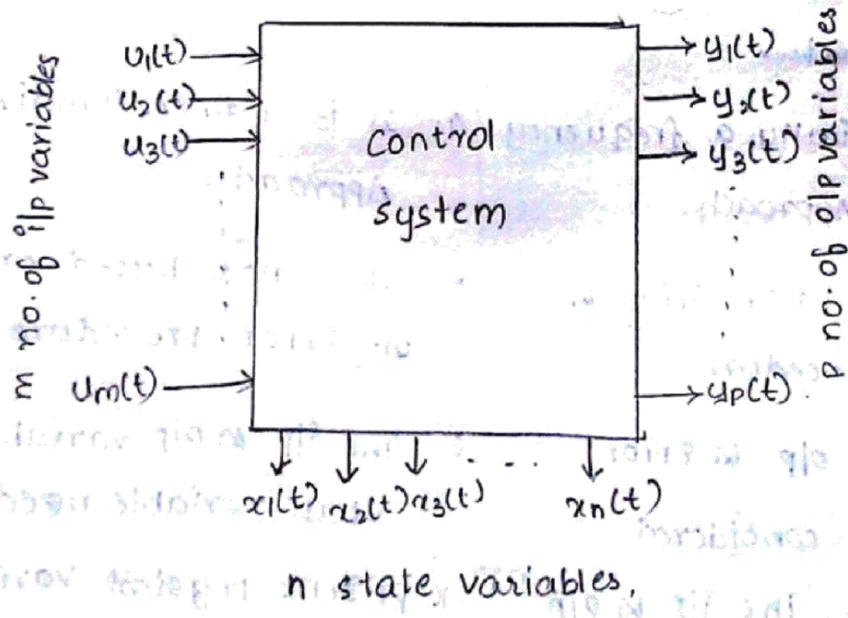


Fig : State Space Representation of a System

Different variables may be represented by the vectors.

Input vector $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$; Output vector $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

State Equations :

State variable Representation can be arranged in the form of 'n' no. of 1st order differential eqns as shown below.

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, \ldots x_n; u_1, u_2, \ldots u_m)$$

'n' no. of differential equations may be written in vector notations as

$$\dot{x}(t) = f[x(t), u(t)]$$

- The set of all possible values which the i/p vector $u(t)$ can have at a time t forms the i/p space of the system
- The set of all possible values which the o/p vector $y(t)$ can have at a time t forms the o/p space of the system
- The set of all possible values which the state vector $x(t)$ can have at a time t forms the state space of system

State model of Linear system :

- The state model of a system consists of state eqn & o/p equation.
- The state eqn of a system is a function of state variables & i/p's.
- For linear time invariant systems, the first derivatives

of state variables can be expressed as Linear combination of state variables and inputs.

$$\dot{x}_1 = (a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n) + (b_{11}u_1 + b_{12}u_2 + \dots b_{1m}u_m)$$

$$\dot{x}_2 = (a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n) + (b_{21}u_1 + b_{22}u_2 + \dots b_{2m}u_m)$$

$$\vdots$$

$$\dot{x}_n = (a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n) + (b_{n1}u_1 + b_{n2}u_2 + \dots b_{nm}u_m)$$

where coefficients a_{ij} & b_{ij} are constants

In the matrix form, above eqns can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\dot{X}(t) = AX(t) + BU(t) \longrightarrow ①.$$

where $X(t)$ is state vector of order $n\times1$

$U(t)$ is input vector of order $m\times1$.

A = system matrix with order $n\times n$

B = Input matrix with order $n\times m$.

Eqn-① is called <u>STATE EQUATION</u> of LTI system.

* The o/p at any time are functions of state variables and input.

∴ Output vector $y(t) = f(x(t), u(t))$

The o/p variables can be expressed as Linear combination of state variables and inputs.

$$y_1 = (c_{11}x_1 + c_{12}x_2 + \dots c_{1n}x_n) + (d_{11}u_1 + d_{12}u_2 + \dots d_{1m}u_m)$$

$$y_2 = (c_{21}x_1 + c_{22}x_2 + \dots c_{2n}x_n) + (d_{21}u_1 + d_{22}u_2 + \dots d_{2m}u_m)$$

$$\vdots$$

$$y_p = (c_{p1}x_1 + c_{p2}x_2 + \dots c_{pn}x_n) + (d_{p1}u_1 + d_{p2}u_2 + \dots d_{pm}u_m)$$

In matrix form, above Eqns can be written as.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$Y(t) = CX(t) + DU(t) \longrightarrow ②$$

where $X(t)$ = State vector of order $n \times 1$

$U(t)$ - Input vector of order $m \times 1$

$Y(t)$ = output vector of order $p \times 1$

C = output matrix of order $p \times n$

D = Transmission matrix of order $p \times m$.

Eqn - ② is called OUTPUT EQUATION of LTI system.

The state Eqn & o/p Eqn together called as state model of the system. Hence the state model of the LTI system is given by.

$$\dot{X}(t) = AX(t) + BU(t) \quad \text{— state Eqn}$$
$$Y(t) = CX(t) + DU(t) \quad \text{— o/p Eqn.}$$

State Diagram :

The pictorial representation of state model of the system is called State Diagram. State Diagram of a system can be either in block diagram form or in signal flow graph form.

- State Diagrams describe the relationship among state variables & provides information physical interpretation of state variables

- Time Domain state diagram can be obtained from Diff. eqns. S-domain state diagram can be obtained from Transfer function of system.

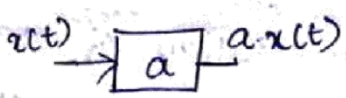
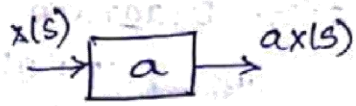
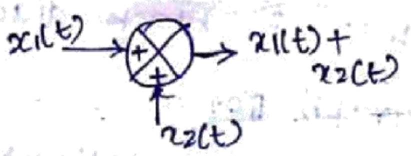
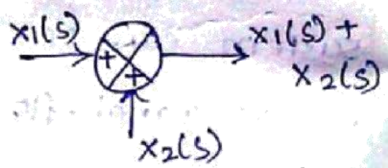
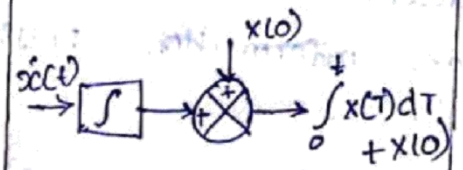
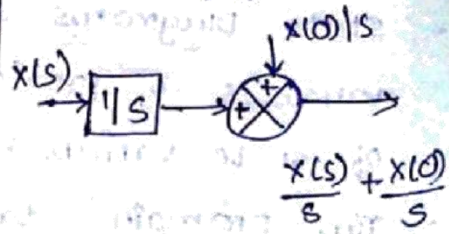
- State Diagram provides relation b/w Time domain & s-domain.
- State Diagram of state model is constructed by using 3- basic elements - scalar, adder & Integrator

Scalar :-
It is used to multiply a signal by a constant.

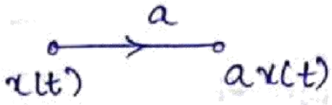
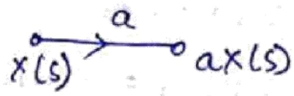
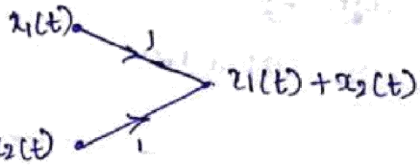
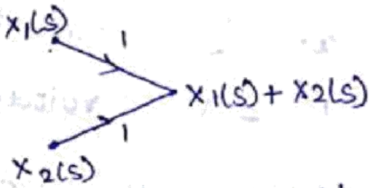
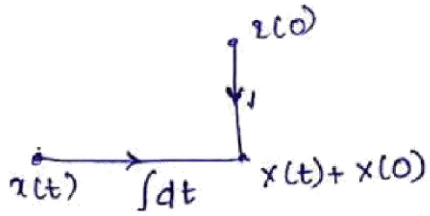
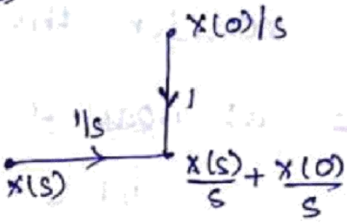
Adder :
It is used to add two or more signals.

Integrator :
It is used to Integrate the signal. They are used to Integrate the derivatives of state variables to get state variables.
- The Initial conditions of the state variable can be added by using an adder after Integration.

* Elements of Block Diagram :

Element	Time Domain	s - Domain
Scalar		
Adder		
Integrator		

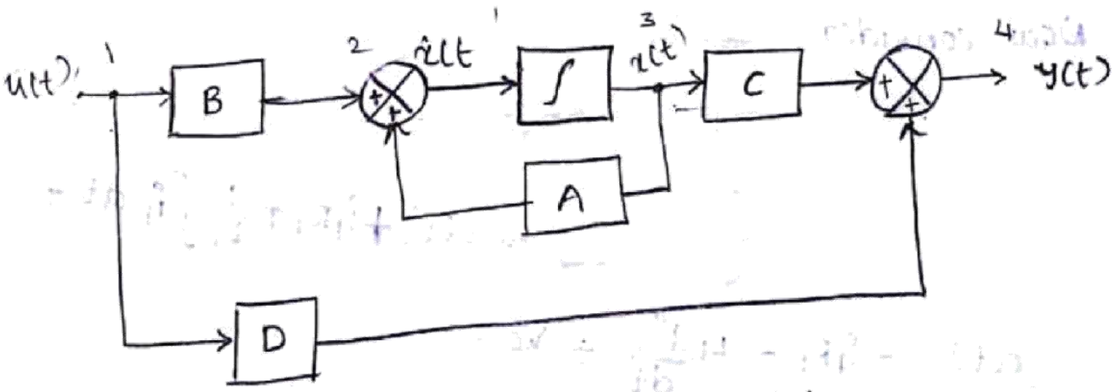
Elements of Signal Flow Graph :

Element	Time Domain	S- Domain
Scalar		
Adder		
Integrator		

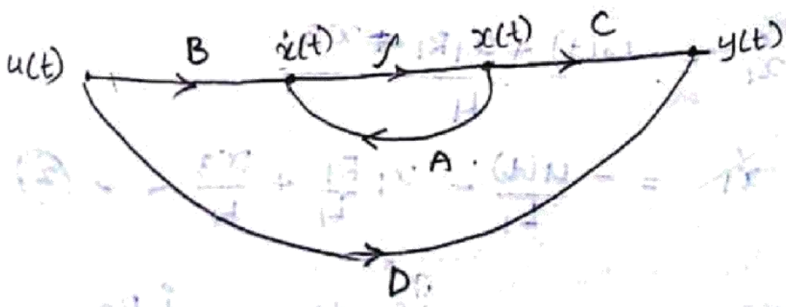
State Eqn : $\dot{x}(t) = A\,x(t) + B\,U(t)$

output Eqn : $Y(t) = C\,x(t) + D\,U(t)$

Block Diagram of State model :

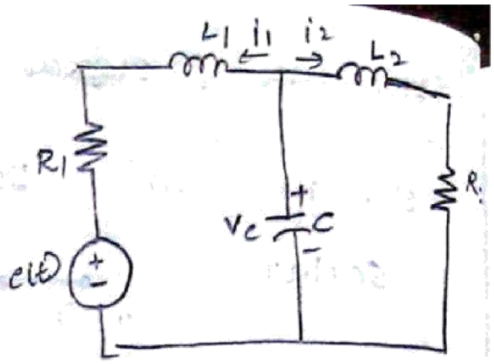


Signal flow graph of State Model :



1. Obtain the state model of Electrical network shown in figure by choosing minimum no. of state variables.

Sol:- Let 3 state variables x_1, x_2, x_3 are related to physical quantities.

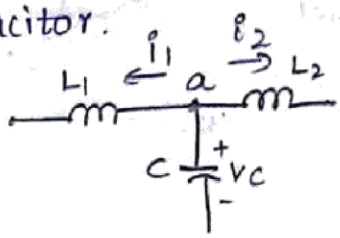


$x_1 = i_1$ = current through L_1.

$x_2 = i_2$ = current through L_2

$x_3 = V_c$ = voltage across capacitor.

consider the node



at node-a.

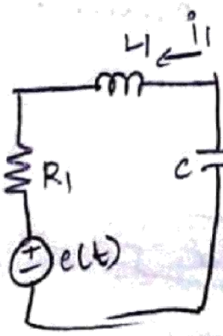
$$i_1 + i_2 + C\frac{dV_c}{dt} = 0$$

$$\Rightarrow x_1 + x_2 + C\frac{dx_3}{dt} = 0$$

$$\Rightarrow x_1 + x_2 + C\dot{x}_3 = 0 \Rightarrow \dot{x}_3 = \frac{-(x_1+x_2)}{C}$$

$$\dot{x}_3 = -\frac{x_1}{C} - \frac{x_2}{C} \rightarrow (1)$$

Now consider



~~$\Rightarrow e(t) + i_1R_1 + \frac{1}{L_1}\int i_1\,dt + \cdot$~~

$$e(t) = -i_1R_1 - L_1\frac{di_1}{dt} + V_c.$$

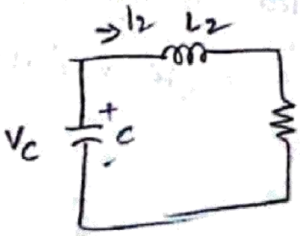
Let $e(t) = u(t)$

$$\Rightarrow u(t) = -x_1R_1 - L_1\dot{x}_1 + x_3$$

$$\Rightarrow -\dot{x}_1 = \frac{u(t) + x_1R_1 - x_3}{L_1} \quad \Rightarrow \dot{x}_1 = -\frac{x_1R_1}{L} + \frac{x_3}{L_1} - \frac{u(t)}{L}$$

$$\Rightarrow \dot{x}_1 = -\frac{u(t)}{L_1} - x_1\frac{R_1}{L_1} + \frac{x_3}{L_1} \rightarrow (2)$$

Now consider



$$V_c = \frac{di_2}{dt}L_2 + i_2R_2$$

$$\Rightarrow x_3 = \dot{x}_2L_2 + x_2R_2.$$

$$\Rightarrow \dot{x}_2 = -\frac{x_2R_2}{L_2} + \frac{x_3}{L_2} \rightarrow (3)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & \frac{1}{L_1} \\ 0 & -R_2/L_2 & 1/L_2 \\ -\frac{1}{C} & -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -1/L_1 \\ 0 \\ 0 \end{bmatrix} [u] .$$

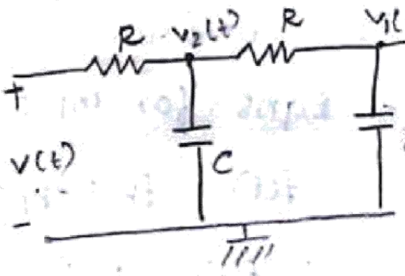
↳ state Equation

$y_1 = i_1 R_1 = x_1 R_1$ → Assume voltage across & current through R_2 are o/p variables y_1 & y_2 resp ∴ The o/p eqns are given by $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & R_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ → Output Equation.

$y_2 = i_2 R_2 = x_2 R_2$

$$\Rightarrow \cancel{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

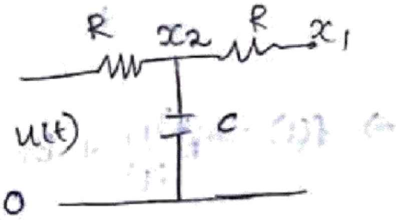
2. Obtain state model for the Electrical network by choosing $v_1(t)$ & $v_2(t)$ as state variables & $v(t)$ ~~as~~ $u(t)$



Sol:- Apply KCL at node - 1

Let $V_1(t) = x_1$, $V_2(t) = x_2$.

$\Rightarrow$ ~~$V_2(t) - V(t)$~~

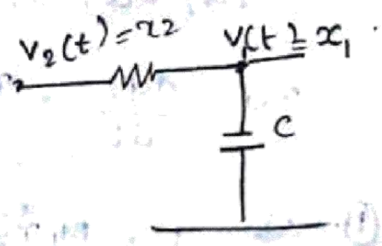


$$\Rightarrow \frac{x_2 - u(t)}{R} + \frac{x_2 - x_1}{R} + C\frac{dx_2}{dt} = 0$$

$$\Rightarrow x_2\left[\frac{2}{R}\right] - \frac{x_1}{R} + C\dot{x}_2 - \frac{u(t)}{R} = 0$$

$$\Rightarrow \dot{x}_2 = \frac{x_1}{RC} - x_2\left[\frac{2}{RC}\right] + \frac{u(t)}{RC} \longrightarrow (1)$$

Now consider



Apply KCL

$$C\frac{dx_1}{dt} + \frac{x_1 - x_2}{R} = 0$$

$$\frac{x_1}{R} + C\dot{x}_1 - \frac{x_2}{R} = 0.$$

$$\dot{x}_1 = -\frac{x_1}{RC} + \frac{x_2}{RC} \longrightarrow (2).$$

matrix form

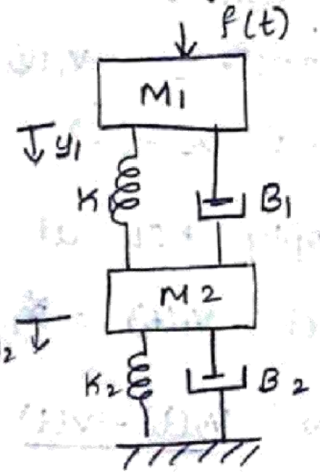
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/RC & +1/RC \\ +1/RC & -2/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ +1/RC \end{bmatrix} [u]$$

$\rightarrow$ State Equation

$y_1(t) = V_1(t) = x_1$.

Matrix form $y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\rightarrow$ output Equation.

3. construct the state model of mechanical system shown below.



Sol:- Eqns for M_1 :

$f(t) = f_{K_1} + f_{b_1} + f_{M_1}$

$$\Rightarrow f(t) = M\frac{d^2y_1}{dt^2} + B_1\frac{d(y_1-y_2)}{dt} + K_1(y_1-y_2)$$

$$\Rightarrow f(t) = M\frac{d^2y_1}{dt^2} + B_1\frac{dy_1}{dt} - B_1\frac{dy_2}{dt} + K_1y_1 - K_1y_2 \rightarrow (1)$$

Let Eqns for M_2 :

$$0 = f_{K_2} + f_{b_2} + f_{K_1} + f_{b_1} + f_{M_2}$$

$$0 = K_2y_2 + B_2\frac{dy_2}{dt} + K_1(y_2-y_1) + B_1\frac{d}{dt}(y_2-y_1) + M_2\frac{d^2y_2}{dt^2} \quad \text{— (2)}$$

Let $f(t) = u$, $x_1 = y_1$, $x_2 = y_2$, $x_3 = \frac{dy_1}{dt}$,

$x_4 = \frac{dy_2}{dt}$, $\dot{x}_3 = \frac{d^2y_1}{dt^2}$, $\dot{x}_4 = \frac{d^2y_2}{dt^2}$

(1) $\Rightarrow$ $U = M_1\dot{x}_3 + B_1x_3 - B_1x_4 + K_1x_1 - K_1x_2$.

$U = M_1\dot{x}_3 + B_1(x_3 - x_4) + K_1(x_1 - x_2)$.

$$② \Rightarrow \quad M_2\dot{x}_4 + B_2 x_4 + B_1(x_4 - x_3) + K_1(x_2 - x_1) + K_2x_2 = 0$$

$$① \Rightarrow \quad \dot{x}_3 = \frac{u - B_1(x_3 - x_4) + K_1(x_1 - x_2)}{M_1} \rightarrow$$

$$② \Rightarrow \quad \dot{x}_4 = \frac{-[B_2x_4 + B_1(x_4 - x_3) + K_1(x_2 - x_1) + K_2x_2]}{M_2}$$

$$\Rightarrow \dot{x}_3 = -\frac{K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{B_1}{M_1}x_4 + \frac{u}{M_1} \rightarrow ③.$$

$$\dot{x}_4 = -\left[\frac{+K_1}{M_2}x_1 + \frac{(K_1+K_2)}{M_2}x_2 - \frac{B_1}{M_2}x_3 + \frac{(B_1+B_2)}{M_2}x_4,\right]$$

$$\Rightarrow \dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4 \rightarrow ④$$

we have

$$x_1 = y_1 \Rightarrow \dot{x}_1 = \frac{dy_1}{dt} = x_3. \rightarrow ⑤$$

$$x_2 = y_2 \Rightarrow \dot{x}_2 = \frac{dy_2}{dt} = x_4. \rightarrow ⑥$$

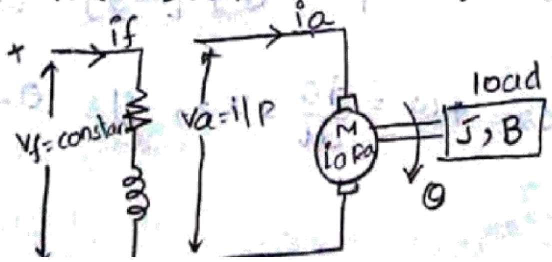
Matrix form of eqns ⑤, ⑥, ③ & ④.

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1/M_1 & K_1/M_1 & \frac{-B_1}{M_1} & \frac{B_1}{M_1} \\ K_1/M_2 & \frac{-(K_1+K_2)}{M_2} & \frac{B_1}{M_2} & \frac{-(B_1+B_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ 0 \end{bmatrix} [u]$$

$$y_1 = x_1, \; y_2 = x_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

4. Determine the state model of Armature control DC motor



$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \longrightarrow ①.$$

$$e_b \propto \frac{d\theta}{dt}, \quad e_b = K_b \frac{d\theta}{dt}.$$

$$T \propto i_a, \quad T = K_t i_a$$

$$\Rightarrow J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \pm T \longrightarrow ②.$$

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + K_b \frac{d\theta}{dt}. \rightarrow ①$$

$$K_t i_a = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \;-\; ②$$

Let. $i_a = x_1, \; \omega = x_2 = \frac{d\theta}{dt}, \; \theta = x_3, \; V_a = U. \Rightarrow \dot{x}_3 = x_2$ $\rightarrow ⑤$

$$① \Rightarrow U = R_a x_1 + L_a \dot{x}_1 + K_b x_2.$$

$$② \Rightarrow K_t x_1 = J\dot{x}_2 + B_1 x_2,$$

$$\Rightarrow \dot{x}_1 = \frac{U - R_a x_1 - K_b x_2}{L_a}.$$

$$\Rightarrow \dot{x}_1 = -\frac{R_a}{L_a}x_1 - \frac{K_b}{L_a}x_2 + \frac{U}{L_a} \rightarrow ③$$

$$\dot{x}_2 = \frac{K_t x_1 - B_1 x_2}{J}$$

$$\Rightarrow \dot{x}_2 = \frac{K_t}{J}x_1 - \frac{B_1}{J}x_2 \longrightarrow ④.$$

Matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-R_a/L_a & -K_b/L_a & 0\\ K_t/J & -B_1/J & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}\frac{1}{L_a}\\ 0\\ 0\end{bmatrix}[U]$$

$$y_1 = i_a = x_1, \quad y_2 = \omega = \frac{d\theta}{dt} = x_2, \quad y_3 = \theta = x_3.$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

* State space Representation using phase variables:

Phase variables are defined as those particular variables which are obtained from one of the system variables and its derivatives.

Usually, the variable used is system output. The state model using phase variable can be easily determined if the system model is already known in the Differential Equation (or) Transfer function form.

Advantages :

- It can be directly formed by Inspection from differential Equations governing the system.
- It provides a link between Transfer function approach and time domain design approach.

Disadvantages :

- Phase variables are not physical variables of the system and therefore not available for measurement and control purpose.

1. Construct a state model for a system characterised by Differential Eqn $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$..

Sol:- The system order is 3.

∴ The no. of state variables are 3.

Let $x_1 = y$, $x_2 = \frac{dy}{dt} = \dot{x}_1$, $x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$, $x_4 = \frac{d^3y}{dt^3} = \dot{x}_3$

Eqn $\Rightarrow$ $\dot{x}_3 + 6\dot{x}_2 + 11\dot{x}_1 + 6\dot{x}_1 + u = 0$.

$\Rightarrow \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$

$$\Rightarrow \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - u.$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_1 = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u].$$

O/p Eqn :

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

2. $$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

Soln:-

$$Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U[s]$$

$$s^3 = \frac{d^3y}{dt^3}, \quad s^2 = \frac{d^2}{dt^2}, \quad s = \frac{d}{dt}.$$

$$\Rightarrow \frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y - 10u(s) = 0.$$

Let $x_1 = y$, $x_2 = \frac{dy}{dt} = \dot{x}_1$, $x_3 = \frac{d^2y}{dt^2} = \dot{x}_2$, $x_4 = \frac{d^3y}{dt^2} = \dot{x}_3$

$$\Rightarrow \dot{x}_3 + 4x_3 + 2x_2 + x_1 - 10u = 0$$

$$\Rightarrow \dot{x}_3 = -(4x_3 + 2x_2 + x_1 - 10u)$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_1 = x_2.$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ +10 \end{bmatrix} [u]$$

o/p eqn :

$y = x_1$.

$$y = [1\ 0\ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3. Obtain the state variable formulation of a field-controlled DC motor.

Sol:- consider equations for field controlled DC Motor.

$$J\frac{d^2\theta}{dt} + B\frac{d\theta}{dt} = T.$$

$$T \propto I_f \Rightarrow T = K_t I_f$$

$$\Rightarrow J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_t I_f \rightarrow ①.$$

$$L_f\frac{di_f}{dt} + R_f i_f = e_f \longrightarrow ②.$$

Let $i_f = x_1,\ x_2 = \frac{d\theta}{dt} = \omega,\ \theta = x_3,\ e_f = U,\ \cancel{x_3 = \dot{x}_2} \rightarrow ③$

$$① \Rightarrow J\dot{x}_2 + Bx_2 = K_t x_1$$

$$J\dot{x}_2 + Bx_2 - K_t x_1 = 0$$

$$\dot{x}_2 = \frac{K_t x_1}{J} - \frac{B}{J}x_2 \longrightarrow ④.$$

$$② \Rightarrow L_f\dot{x}_1 + R_f x_1 = U$$

$$\dot{x}_1 = \frac{U}{L_f} - \frac{x_1 R_f}{L_f} \longrightarrow ⑤.$$

$$x_2 = \dot{x}_3 \rightarrow ③$$

matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -R_f/L_f & 0 & 0 \\ K_t/J & -B/J & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1/L_f \\ 0 \\ 0 \end{bmatrix} U.$$

Output Eqn..

$$y = x_2 = \frac{d\theta}{dt} = \omega.$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4 Obtain state model for the system whose transfer function is given as $\frac{Y(s)}{U(s)} = \frac{5s+6}{s^2+2s+3}$

Sol:-

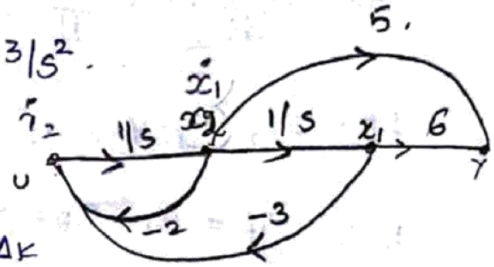
$$\frac{Y(s)}{U(s)} = \frac{5s+6}{s^2+2s+3}.$$

Divide numerator & Denominator with s^2

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{5}{s} + \frac{6}{s^2}}{1 + \frac{2}{s} + \frac{3}{s^2}} = \frac{\frac{5}{s} + \frac{6}{s^2}}{1 - \left[-\frac{2}{s} - \frac{3}{s^2}\right]} = \frac{P_K \Delta_K}{\Delta}$$

Denominator : $1 - \left[-\frac{2}{s} - \frac{3}{s^2}\right]$.

loop $l_1 = -2/s$, loop $l_2 = -3/s^2$.



Numerator : $\frac{5}{s} + \frac{6}{s^2} = \sum_K P_K \Delta_K$

$\Delta_1 = \Delta_2 = 1$ [$\because$ there are no non-touching loops)

$P_1 = \frac{5}{s}$, $P_2 = \frac{6}{s^2}$ [forward path gains)

Equations from signal flow graph

$$\dot{x}_2 = -3x_1 - 2x_2 + U$$

$$\dot{x}_1 = x_2$$

Matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

5. $T(s) = \dfrac{s^2+3s+3}{s^3+2s^2+3s+1}$

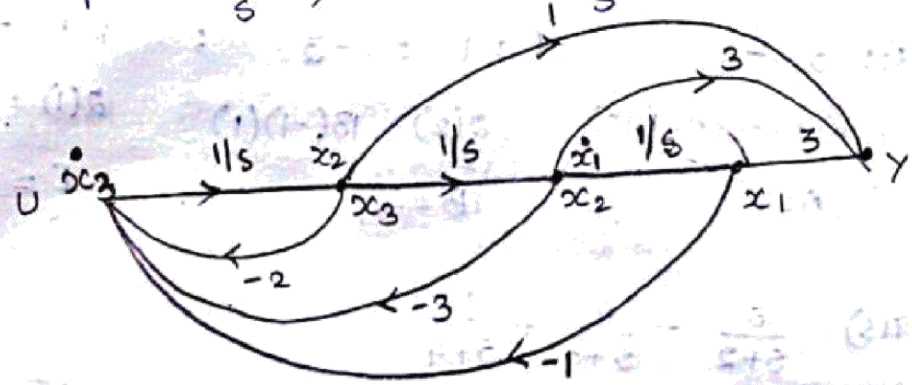
Divide num & den by s^3

$$\Rightarrow T(s) = \frac{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}}{1+\frac{2}{s}+\frac{3}{s^2}+\frac{1}{s^3}} = \frac{\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}}{1-\left[-\frac{2}{s}-\frac{3}{s^2}-\frac{1}{s^3}\right]}$$

Denominator :

$$1-\left[-\frac{2}{s}-\frac{3}{s^2}-\frac{1}{s^3}\right]$$

loop $l_1 = -\frac{2}{s}$, loop $l_2 = -\frac{3}{s^2}$, loop $l_3 = -\frac{1}{s^3}$



Numerator : $\frac{1}{s}+\frac{3}{s^2}+\frac{3}{s^3}$

$\Delta_1 = \Delta_2 = 1.$

$P_1 = \frac{1}{s}$, $P_2 = \frac{3}{s^2}$, $P_3 = \frac{3}{s^3}$.

$\dot{x}_3 = -2x_3 - 3x_2 - x_1 + U$.

$\dot{x}_2 = x_3$

$\dot{x}_1 = x_2$.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}U$$

* STATE SPACE REPRESENTATION USING CANONICAL VARIABLES :

1. Determine the canonical state model of the system whose transfer function is $T(s) = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$

Sol:- $T(s) = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}$.

$2(s+5) = A(s+3)(s+4) + B(s+2)(s+4) + C(s+2)(s+3)$

Let $s = -2$. ; Let $s = -3$. ; Let $s = -4$

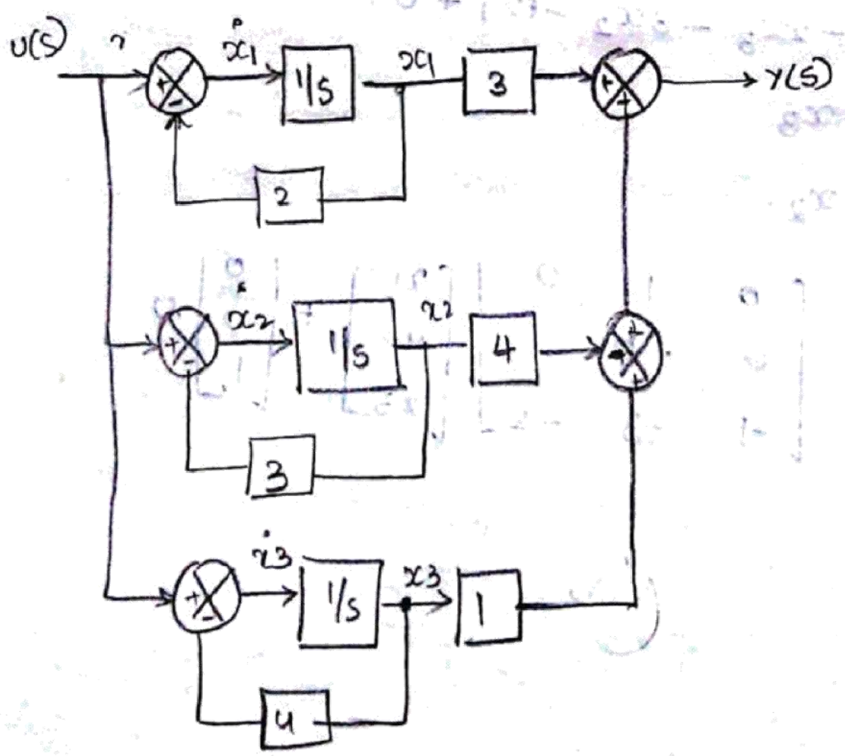
$\Rightarrow 2(3) = A(1)(2)$; $2(2) = B(-1)(1)$; $2(1) = C(-2)(-1)$

$\Rightarrow A = 3$. ; $B = -4$; $C = 1$.

$\Rightarrow T(s) = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4}$

$\frac{Y(s)}{U(s)} = \frac{3}{s\left[1+\frac{2}{s}\right]} - \frac{4}{s\left[1+\frac{3}{s}\right]} + \frac{1}{s\left[1+\frac{4}{s}\right]}$.

$\Rightarrow Y(s) = \frac{3/s}{1+\frac{1}{s}\times 2} \cdot U(s) - \frac{4/s}{1+\frac{1}{s}\times 3} U(s) + \frac{1/s}{1+\frac{1}{s}\times 4}$.



$\dot{x}_1 = -2x_1 + u$ $\quad\quad\quad$ $Y = 3x_1 - 4x_2 + x_3$

$\dot{x}_2 = -3x_2 + u$

$\dot{x}_3 = -4x_3 + u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

* SOLUTION OF HOMOGENEOUS STATE EQUATIONS (SOLUTION OF STATE EQUATION WITHOUT INPUT AND EXCITATION) :

$$x(t) = \left[I + At + \frac{1}{2!} A^2t^2 + \frac{1}{3!} A^3t^3 + \cdots \right] x_0$$

$$x(t) = e^{At} x_0$$

The matrix e^{At} is called state Transition matrix and denoted by $\phi(t)$.

From the solution of the state equation, it is observed that the initial state x_0 at $t=0$ is driven by $x(t)$ at time t by state transition matrix.

* COMPUTATION OF STATE TRANSITION MATRIX :

State Transition Matrix can be computed by using following methods

1. Infinite series method
2. By Laplace Transformation
3. Using Cayley - Hamelton Theorem
4. By canonical Transformation.
5. By sylvesters method.

* COMPUTATION OF STATE TRANSITION MATRIX BY INFINITE SERIES METHOD :

1. Compute state transition matrix by using Infinite series method. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Sol:- $\phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= [1 + 1(0)] + \{1\} = 2.$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}t + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\frac{t^3}{3!}.$$

$$\phi(t) = \begin{bmatrix} 1+t+\frac{t^2}{2!}+\frac{t^3}{3!} & t+\frac{2t^2}{2!}+\frac{3t^3}{3!} \\ 0 & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

* Properties of state transition matrix (STM) & its Definition :

2. COMPUTATION OF STATE TRANSITION MATRIX BY LAPLACE TRANSFORMATION.

1. consider a matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$e^{At} = \phi(t) = L^{-1}[SI - A]^{-1}$$

Sol:- $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$SI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ +2 & s+3 \end{bmatrix}$$

$$[SI - A]^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{s^2+3s+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$L^{-1}\left[\frac{s+3}{(s+1)(s+2)}\right] = \frac{A}{s+1}$$

$$\phi(s) = SI - A = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{A_1}{s+1} + \frac{B_1}{s+2} & \frac{A_2}{s+1} + \frac{B_2}{s+2} \\ \frac{A_3}{s+1} + \frac{B_3}{s+2} & \frac{A_4}{s+1} + \frac{B_4}{s+2} \end{bmatrix}.$$

$s+3 = A_1(s+2) + B_1(s+1)$

$4 = A_1(2)$

$A_1 = 2.$

$1 = B(-1)$

$B_1 = -1.$

$1 = A_2(s+2) + B_2(s+1)$

$A_2 = 1, \quad B_2 = -1$

$-2 = A_3(s+2) + B_3(s+1)$

$A_3 = -2, \quad B_3 = 2$

$s = A_4(s+2) + B_4(s+1)$

$A_4 = -1, \quad B_4 = 2.$

$$\Rightarrow \phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix},$$

$$L^{-1}(\phi(s)) = \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \phi(t) \; e^{At}$$

3. Computation of STM by canonical Transformation,

1. A linear time invariant system is described by following state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} [u] \text{ and}$$

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Sol:- 1. Transform this state model into canonical state model. Also compute STM e^{At}

Step-1 : Finding Eigen values.

$$|\lambda I - A| = 0$$

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ +6 & +11 & \lambda+6 \end{bmatrix}$$

$$|\lambda I - A| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & 6\lambda+6 \end{vmatrix} = 0.$$

$$\lambda[6\lambda^2 - 11] + 1[+6] = 0$$

$$6\lambda^3 - 11\lambda - 6 = 0.$$

$$\lambda[(\lambda^2 + 6\lambda) + 11] + 1[+6] = 0$$

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$\lambda = -1, -2, -3$.

Step-2 :

Finding modal matrix, M. $M = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \quad \lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3.$$

Step-3 :

Finding M^{-1} : $M^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$

Step-4 :

Finding canonical form of state model. i.e.,

Grammian Matrix :. $M^{-1}AM = \tilde{A}$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Step-5 :

$$\tilde{B} = M^{-1}B = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\tilde{C} = CM = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Canonical form of state model.

$$\dot{z} = \tilde{A}z + \tilde{B}U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} z + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} U = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix}$$

$$y = \tilde{C}z + \tilde{D}u$$

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

step: 5 : finding e^{At}.

$$e^{At} = Me^{\Lambda} \cdot M^{-1} \quad \text{where } e^{\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix}$$

$$e^{\Lambda} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}.$$

$$\Rightarrow e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-2t} & e^{-3t} \\ -e^{-t} & -2e^{-2t} & -3e^{-3t} \\ e^{-t} & 4e^{-2t} & 9e^{-3t} \end{bmatrix} \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 3e^{-t} - 3e^{-2t} + e^{-3t} & 2.5e^{-t} - 4e^{-2t} + 1.5e^{-3t} & 0.5e^{-t} - e^{-2t} + 0.5e^{-3t} \\ -3e^{-t} + 6e^{-2t} - 3e^{-3t} & 2.5e^{-t} + 8e^{-2t} - 4.5e^{-3t} & -0.5e^{-t} + 2e^{-2t} + 1.5e^{-3t} \\ 3e^{-t} - 12e^{-2t} + 9e^{-3t} & 2.5e^{-t} - 16e^{-2t} + 13.5e^{-3t} & 0.5e^{-t} - 4e^{-2t} + 4.5e^{-3t} \end{bmatrix}$$

* A Linear Time Invariant system is characterised by homogeneous state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. compute the solution of homogeneous eqn assuming initial vector

$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Sol:- $e^{At} = \phi(t) = L^{-1}[sI - A]^{-1}$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}.$$

$$[sI - A]^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} = \begin{bmatrix} 1/s-1 & 0 \\ 1/(s-1)^2 & 1/s-1 \end{bmatrix}$$

$$e^{At} = \phi(t) = L^{-1}(sI-A)^{-1} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

The solution of the state equation is

$$x(t) = e^{At} x_0 = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

* Compute the solution of given non-homogeneous state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ where u is a unit step function. Assume initial conditions $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Sol:- $X(s) = \phi(s)[X(0) + BU(s)]$ $\quad x(t) = L^{-1}(X(s))$

$$\phi(s) = [sI - A]^{-1} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/s-1 & 0 \\ 1/(s-1)^2 & 1/s-1 \end{bmatrix}.$$

$$X(s) = \begin{bmatrix} 1/s-1 & 0 \\ 1/(s-1)^2 & 1/s-1 \end{bmatrix} \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s} \right]$$

$$= \begin{bmatrix} 1/s-1 & 0 \\ 1/(s-1)^2 & 1/s-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/s \end{bmatrix} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{(s-1)^2} + \frac{1}{s(s-1)} \end{bmatrix}.$$

$$L^{-1}\left[\frac{1}{s(s-1)}\right] = \frac{A}{s} + \frac{B}{s-1}$$

$$1 = A(s-1) + Bs.$$

$$A = -1 \;.\; B = 1$$

$$\Rightarrow L^{-1}\left[\frac{1}{s(s-1)}\right] = L^{-1}\left[\frac{-1}{s} + \frac{1}{s-1}\right]$$

$$\Rightarrow \quad x(t) = \begin{bmatrix} e^t \\ te^t - 1 + e^t \end{bmatrix}$$

$$\therefore \; x(t) = L^{-1} X(s) = \begin{bmatrix} e^t \\ e^t(t+1) - 1 \end{bmatrix}$$

* Transfer function from state model:

1. Obtan T.F if $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$,

$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Sol:- $T(s) = C[sI - A]^{-1} B$.

$$[sI - A]^{-1} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1} = \frac{1}{(s+5)(s+1)+3}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+5+3}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+8}\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1}{(s^2+6s+8)} & \frac{-1}{s^2+6s+8} \\ \frac{3}{s^2+6s+8} & \frac{s+5}{s^2+6s+8} \end{bmatrix}$$

$$C[sI - A]^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{s+1}{(s+2)(s+4)} & \frac{-1}{(s+2)(s+4)} \\ \frac{3}{(s+2)(s+4)} & \frac{s+5}{(s+2)(s+4)} \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \left[\frac{s+1+6}{(s+2)(s+4)} \quad ; \quad \frac{-1+2s+10}{(s+2)(s+4)}\right]\begin{bmatrix}2\\5\end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)}\begin{bmatrix}s+7 & 2s+9\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}.$$

$$= \frac{1}{(s+2)(s+4)}(2s+14+10s+45)$$

$$= \frac{12s+59}{(s+2)(s+4)}$$

2. Obtain T.F if $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u$ & $y = [1 \ 0]\begin{bmatrix}x_1\\x_2\end{bmatrix}$

Sol:- $T(s) = C[sI-A]^{-1}B$.

$$sI-A = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix} = \begin{bmatrix}s & -1\\2 & s+3\end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix}s+3 & 1\\-2 & s\end{bmatrix}$$

$$= \frac{1}{s^2+3s+2}\begin{bmatrix}s+3 & 1\\-2 & s\end{bmatrix}$$

$$C[sI-A]^{-1}B = [1 \quad 0]\frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\\-2 & s\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}.$$

$$= \frac{1}{(s+1)(s+2)}(1). = \frac{1}{(s+1)(s+2)}$$

* Controllability of Linear Systems :

A system is said to be completely state controllable at time t_0, if it is possible by means of an unconstrained control vector u(t) to transfer the system from an initial state $x(t_0)$ to any other desired state in a finite interval of time.

For a linear time invariant system described by the dynamic equations. $\dot{x}(t) = Ax(t) + Bu(t)$ & $y(t) = Cx(t) + Du$

where $x(t)$ is an $n \times 1$ state vector & $u(t)$ is a scalar to be completely state controllable. It is necessary & sufficient that the following $n \times n$, controllability matrix has a rank of n.

$$S = [B \quad AB \quad A^2B \; \cdots \; A^{n-1}B]$$

since the matrix A and B are involved, sometimes we say that the pair [A,B] is controllable which implies that s is of rank n.

1. Evaluate $\dot{x} = A$ controllability of system $\dot{x} = Ax + Bu$, $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x(t)$ is n-dimensional state vector.

Sol:- $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}_{2 \times 2} \Rightarrow n = 2$.

controllability matrix $S = [B \quad AB]$.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$|S| = 0$, Rank $= 1$

$\because$ $|S| \neq 0$ & Rank $\neq n$ $\Rightarrow$ the given system is not controllable.

2. $\dot{x} = Ax + Bu$, $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

Sol:- $n = 2$, $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

controllable matrix $S = [B \quad AB]$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|S| = -1 \quad , \quad \text{rank} = 2$$

$\Rightarrow$ system is controllable.

3. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Sol:- $S = [B \quad AB \quad A^2B]$.

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -3 & 0 \\ 7 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & 0 & -3 & 0 & 7 & 0 \end{bmatrix}$$

$$|S| = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -3 \\ -3 & 0 & 7 \end{bmatrix} = 1(0) = 0.$$

$$|S| = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{bmatrix} = 1 \neq 0.$$

Rank = 3 = n = order

$\Rightarrow$ The system is controllable. $\because |S| \neq 0$ & Rank = n.

* **Observability:**

A system is said to be completely observable if every state $x(t_0)$ can be completely identified by measurements of outputs $y(t)$ over a finite time interval.

If the system is not completely observable means that few of its state variables are not practically measurable & are shielded from observation.

- similar to controlability, the observability of the system can be obtained by Kalman's Test.

Kalman's Test for Observability:

consider n-th order MIMO (mul. i/p mult. o/p) LTI system, represented by its state equation as $\dot{x} = AX(t) + BU(t)$ & $Y(t) = CX(t)$ where C is $(1\times n)$ matrix; $y(t)$ is $p\times1$ matrix. The system is completely observable if and only if the rank of composite matrix Q_0 is 'n'

The composite matrix Q_0 is given by

$$Q_0 = [C^T;\ A^TC^T \ldots (A^T)^{n-1}C^T]$$ where C^T = Transpose of C.

A^T = Transpose of matrix A.

Thus if Rank of $Q_0 = n$, then the system is completely observable. The matrix Q_0 is called Test matrix for observability.

1. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \quad y(t) = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol:- $$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \quad C = [1 \quad 0], \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$Q_0 = [C^T \quad A^T C^T] = \Big[$

$A^T C^T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow Q_0 = [C^T \quad A^T C^T] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad |Q_0| = 1 \neq 0.$

$R_2 \to$ Rank $= 2 = n$

$\therefore$ The system is observable

2. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [3 \quad 4 \quad 1]$

Sol: $A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \quad C^T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$A^T C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$(A^{T^2}) C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 6 \\ 1 & -3 & -2+9 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -8+6 \\ 3-12+7 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}.$

$Q_0 = [C^T \quad A^T C^T \quad (A^T)^2 C^T]$

$= \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}. \Rightarrow |Q_0| = 3[-2+2] = 0$

Ra

$C_3 \to C_3/-2. \Rightarrow Q_0 = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2 = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 1 \\ -3 & 0 & 0 \end{bmatrix}; \; R_3 \to R_3 + R_1 = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Rank $= 2 \neq n$

$\therefore$ The system is not observable [$\because |Q_0| = 0$ & Rank $\neq n$]

3. Determine state controllability & observability of the system described by $A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Sol: Controlability : $S = [B \;\; AB \;\; A^2B]$

$$AB = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} +8 & -3 & -1 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 0 & 3 \\ 2 & 1 \end{bmatrix}.$$

$$S = \begin{bmatrix} 0 & 1 & 2 & -2 & -2 & 7 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}.$$

$$|S| = \begin{vmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \cancel{1[0]}-1[-4]+2[0] = -4 \neq 0.$$

Rank = 3.

system is controllable.

Observability : $Q_0 = [C^T \;\; A^TC^T \;\; (A^T)^2C^T]$

$$C^T = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$A^TC^T = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$(A^T)^2C^T = \begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 & 0 \\ -3 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 11 \\ 0 & -4 \\ 1 & -1 \end{bmatrix}.$$

$$Q_0 = \begin{bmatrix} 0 & 1 & 0 & -4 & 0 & 11 \\ 0 & 1 & 0 & 1 & 0 & -4 \\ 1 & 0 & 1 & 2 & 1 & -1 \end{bmatrix} \quad |Q_0| = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0.$$

$$|Q_0| = \begin{vmatrix} -4 & 0 & 11 \\ 1 & 0 & -4 \\ 2 & 1 & -1 \end{vmatrix} = -4[4]+11[1] = -5 \neq 0.$$

Rank = 3.

∴ The system is completely observable.